Evolution equations for special Lagrangian 3-folds in $\mathbb{C}^3$

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1 Introduction

This is the third in a series of papers [4, 5, 6, 7, 8] constructing examples of special Lagrangian submanifolds (SL m-folds) in $\mathbb{C}^m$. The principal motivation for these papers is to lay the foundations for a study of the singularities of compact special Lagrangian m-folds in Calabi–Yau m-folds, particularly in low dimensions such as $m = 3$. Special Lagrangian m-folds in $\mathbb{C}^m$ should provide local models for how singularities develop in special Lagrangian m-folds in Calabi–Yau m-folds.

Understanding such singularities will be essential in making rigorous the explanation of Mirror Symmetry of Calabi–Yau 3-folds $X, \hat{X}$ proposed by Strominger, Yau and Zaslow [9], which involves dual 'fibrations' of $X, \hat{X}$ by special Lagrangian 3-tori, with some singular fibres. It will also be important in resolving conjectures made by the author [3], which attempt to define an invariant of Calabi–Yau 3-folds by counting special Lagrangian homology 3-spheres.

In this paper we construct and study several families of special Lagrangian 3-folds in $\mathbb{C}^3$, using the 'evolution' construction method of [5]. In [5, §3–§4] we described a general construction of SL m-folds in $\mathbb{C}^m$, which will be summarized in §3 below. The construction requires a set of *evolution data* (P, χ), including an $(m-1)$-submanifold P in $\mathbb{R}^n$ for $n \geqslant m$. Then N is the subset of $\mathbb{C}^m$ swept out by the image of P under a 1-parameter family of linear or affine maps $\phi_t : \mathbb{R}^n \to \mathbb{C}^m$, which satisfy a first-order, nonlinear o.d.e. in t.

In [5] we restricted our attention to evolution data (P, χ) in which $n = m$ and P is a quadric in $\mathbb{R}^m$. In §4 we shall establish a correspondence between sets of evolution data with $m = 3$ and homogeneous symplectic 2-manifolds Σ with a transitive, faithful, Hamiltonian action of a Lie group G. This enables us to write down various interesting new examples of evolution data with $m = 3$.

We shall focus on two examples in particular, and study the associated SL 3-folds in $\mathbb{C}^3$ in detail. The first, given in Example 4.2, comes from the action of $\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ on $\mathbb{R}^2$ by affine transformations, and the corresponding family of SL 3-folds are discussed in sections 5–11.

The second example, in Example 4.4, comes from the action of $K \ltimes U_k$ on $T^*\mathbb{R}$ for $k \geqslant 1$, where $K = \mathbb{R}_+ \ltimes \mathbb{R}$ is the group of oriented affine transformations

of $\mathbb{R}$, and U_k the vector space of polynomial 1-forms on $\mathbb{R}$ of degree less than k, acting on $T^*\mathbb{R}$ by translation in the fibres. This gives a series of families of ruled SL 3-folds in $\mathbb{C}^3$, which are discussed in §12.

The construction of §5–§11 involves a family of quadratic maps $\Phi_t : \mathbb{R}^2 \to \mathbb{C}^3$ depending on $t \in \mathbb{R}$, which satisfy the o.d.e.

$$\frac{\mathrm{d}\Phi_t}{\mathrm{d}t} = \frac{\partial \Phi_t}{\partial y_1} \times \frac{\partial \Phi_t}{\partial y_2},$$

where (y_1, y_2) are the coordinates on $\mathbb{R}^2$, and '$\times$' is an anti-bilinear cross product on $\mathbb{C}^3$. Defining $\Phi : \mathbb{R}^3 \to \mathbb{C}^3$ by $\Phi(y_1, y_2, t) = \Phi_t(y_1, y_2)$, it turns out that under certain conditions on the initial data Φ_0 the image $N = \operatorname{Image}\Phi$ is a special Lagrangian 3-fold.

For generic initial data this map Φ is an immersion, so that N is a nonsingular immersed 3-submanifold diffeomorphic to $\mathbb{R}^3$. But for a set of initial data of real codimension one, Φ is not an immersion, and N has singular points. Also, under certain circumstances Φ may be periodic in t, and then N will be generically diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$ rather than $\mathbb{R}^3$.

The basic details of the construction are explained in §5, in terms of o.d.e.s for vector-valued functions $\mathbf{z}_1, \ldots, \mathbf{z}_6 : \mathbb{R} \to \mathbb{C}^3$. We discuss the symmetries of the construction, and show that the family of SL 3-folds we have constructed, up to automorphisms of $\mathbb{C}^3$, is 9-dimensional. Section 6 studies and describes the singularities of the corresponding SL 3-folds, which we believe are of a new kind.

The goal of sections 7–11 is to solve the o.d.e.s for $\mathbf{z}_1, \ldots, \mathbf{z}_6$ as explicitly as we are able to, and so to write our examples of SL 3-folds as explicitly as possible. To do this we split into several cases, and use the symmetries of the problem to write each case in a convenient form. In §7 we divide into four cases (i)–(iv) of increasing complexity, depending on the rank of the homogeneous quadratic part of Φ. Cases (i) and (ii) are easy and are dealt with at once. Case (iii) is the subject of §8, and case (iv) is divided into subcases and discussed in §9–§11.

In Theorems 8.4, 10.3 and 11.5 we are able to write down three families of special Lagrangian 3-folds in $\mathbb{C}^3$ very explicitly, and these are some of the main results of the paper. Also, in §11.2 we study the condition for the family $\{\Phi_t : t \in \mathbb{R}\}$ to be periodic in a special case. This has a surprisingly abundant and structured set of solutions, which leads in Theorem 11.6 to a countable set of distinct families of immersed SL 3-folds diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$.

Section 12 then studies the series of families of SL 3-folds arising from the sets of affine evolution data given in Example 4.4. The development follows parts of §5–§11 closely, and so we leave out some of the details. For each $k = 1, 2, \ldots$ we construct a family of immersed SL 3-folds in $\mathbb{C}^3$ diffeomorphic to $\mathbb{R}^3$, which can be written down in parametric form entirely explicitly.

For $k = 1$ they are isomorphic to the SL 3-folds of [5, Ex. 7.5], and for $k = 2$ they are isomorphic to the family studied in §8. When $k \geqslant 3$ these families include many nontrivial periodic solutions, yielding families of immersed SL 3-folds in $\mathbb{C}^3$ diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$.

2 Special Lagrangian submanifolds in $\mathbb{C}^m$

We begin by defining *calibrations* and *calibrated submanifolds*, following Harvey and Lawson [2].

Definition 2.1 Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane* V on M is a vector subspace V of some tangent space T_xM to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k-plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k-form on V.

Now let φ be a closed k-form on M. We say that φ is a *calibration* on M if for every oriented k-plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space T_xN for $x \in N$ is an oriented tangent k-plane. We say that N is a *calibrated submanifold* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [2, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in $\mathbb{C}^m$, taken from [2, §III].

Definition 2.2 Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, and define a metric g, a real 2-form ω and a complex m-form Ω on $\mathbb{C}^m$ by

$$g = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m),$$
$$\text{and} \quad \Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m.$$

Then $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$ are real m-forms on $\mathbb{C}^m$. Let L be an oriented real submanifold of $\mathbb{C}^m$ of real dimension m. We say that L is a *special Lagrangian submanifold* of $\mathbb{C}^m$ if L is calibrated with respect to $\mathrm{Re}\,\Omega$, in the sense of Definition 2.1. We will often abbreviate 'special Lagrangian' by 'SL', and 'm-dimensional submanifold' by 'm-fold', so that we shall talk about SL m-folds in $\mathbb{C}^m$.

As in [3, 4] there is also a more general definition of special Lagrangian submanifolds involving a *phase* $\mathrm{e}^{i\theta}$, but we will not use it in this paper. Harvey and Lawson [2, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds.

Proposition 2.3 *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into an SL submanifold of $\mathbb{C}^m$ if and only if $\omega|_L \equiv 0$ and $\mathrm{Im}\,\Omega|_L \equiv 0$.*

Note that an m-dimensional submanifold L in $\mathbb{C}^m$ is called *Lagrangian* if $\omega|_L \equiv 0$. Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that $\mathrm{Im}\,\Omega|_L \equiv 0$, which is how they get their name.

3 Review of the 'evolution' construction of [5]

We now review the construction of special Lagrangian m-folds in $\mathbb{C}^m$ given by the author in [5, §3], which will be used in §5 and §12 to construct the SL 3-folds we are interested in. There are two versions of the construction, *linear* and *affine* (linear plus constant), and we shall be using the affine version. The construction depends on some *evolution data*, which we now define, following [5, Def. 3.4].

Definition 3.1 Let $2 \leqslant m \leqslant n$ be integers. A set of *affine evolution data* is a pair (P, χ), where P is an $(m-1)$-dimensional submanifold of $\mathbb{R}^n$, and $\chi : \mathbb{R}^n \to \Lambda^{m-1}\mathbb{R}^n$ is an affine map, such that $\chi(p)$ is a nonzero element of $\Lambda^{m-1}TP$ in $\Lambda^{m-1}\mathbb{R}^n$ for each nonsingular $p \in P$. We suppose also that P is not contained in any proper affine subspace $\mathbb{R}^k$ of $\mathbb{R}^n$.

Let $\mathrm{Aff}(\mathbb{R}^n, \mathbb{C}^m)$ be the affine space of affine maps $\phi : \mathbb{R}^n \to \mathbb{C}^m$, and define $\mathcal{C}_P$ to be the subset of $\phi \in \mathrm{Aff}(\mathbb{R}^n, \mathbb{C}^m)$ satisfying

(i) $\phi^*(\omega)|_P \equiv 0$, and

(ii) $\phi|_{T_pP} : T_pP \to \mathbb{C}^m$ is injective for all p in a dense open subset of P.

Then $\mathcal{C}_P$ is nonempty, and is an open set in the intersection of a finite number of quadrics in $\mathrm{Aff}(\mathbb{R}^n, \mathbb{C}^m)$.

We may define *linear evolution data* in the same way, but using linear rather than affine maps. With this definition, the construction is contained in the following theorem, taken from [5, Th. 3.5].

Theorem 3.2 *Let (P, χ) be a set of affine evolution data, and $n, m, \mathrm{Aff}(\mathbb{R}^n, \mathbb{C}^m)$ and $\mathcal{C}_P$ be as above. Suppose $\phi \in \mathcal{C}_P$. Then there exists $\epsilon > 0$ and a unique real analytic family $\{\phi_t : t \in (-\epsilon, \epsilon)\}$ in $\mathcal{C}_P$ with $\phi_0 = \phi$, satisfying the equation*

$$\left(\frac{\mathrm{d}\phi_t}{\mathrm{d}t}(x)\right)^b = (\phi_t)_*(\chi(x))^{a_1 \dots a_{m-1}} (\mathrm{Re}\,\Omega)_{a_1 \dots a_{m-1} a_m} g^{a_m b} \tag{1}$$

for all $x \in \mathbb{R}^n$, using the index notation for tensors in $\mathbb{C}^m$. Furthermore, $N = \{\phi_t(p) : t \in (-\epsilon, \epsilon),\ p \in P\}$ is a special Lagrangian submanifold in $\mathbb{C}^m$ wherever it is nonsingular.

Here is a brief explanation of the theorem and its proof. Equation (1) is a first-order o.d.e. upon ϕ_t. The key point to note is that as χ is affine, the right hand side of (1) is affine in x, and so (1) makes sense as an evolution equation for affine maps ϕ_t. However, the right hand side of (1) is a homogeneous polynomial of order $m-1$ in ϕ_t, so for $m > 2$ it is a *nonlinear* o.d.e.

The special Lagrangian m-fold N is the total space of a 1-parameter family of real $(m-1)$-dimensional submanifolds $\phi_t(P)$ of $\mathbb{C}^m$, each of which is an affine image of the $(m-1)$-manifold P in $\mathbb{R}^n$. Thus we can think of (1) as an evolution equation in a certain class of real $(m-1)$-submanifolds of $\mathbb{C}^m$.

The theorem is proved using the following result, taken from [4, Th. 3.3].

Theorem 3.3 *Let P be a compact, orientable, real analytic $(m-1)$-manifold, χ a real analytic, nonvanishing section of $\Lambda^{m-1}TP$, and $\phi : P \to \mathbb{C}^m$ a real analytic embedding (immersion) such that $\phi^*(\omega) \equiv 0$ on P. Then there exists $\epsilon > 0$ and a unique family $\{\phi_t : t \in (-\epsilon, \epsilon)\}$ of real analytic maps $\phi_t : P \to \mathbb{C}^m$ with $\phi_0 = \phi$, satisfying the equation*

$$\left(\frac{\mathrm{d}\phi_t}{\mathrm{d}t}\right)^b = (\phi_t)_*(\chi)^{a_1 \dots a_{m-1}} (\mathrm{Re}\,\Omega)_{a_1 \dots a_{m-1} a_m} g^{a_m b}, \tag{2}$$

using the index notation for (real) tensors on $\mathbb{C}^m$. Define $\Phi : (-\epsilon, \epsilon) \times P \to \mathbb{C}^m$ by $\Phi(t, p) = \phi_t(p)$. Then $N = \mathrm{Image}\,\Phi$ is a nonsingular embedded (immersed) special Lagrangian submanifold of $\mathbb{C}^m$.

This constructs SL m-folds in $\mathbb{C}^m$ by evolving arbitrary (compact) real analytic $(m-1)$-submanifolds P of $\mathbb{C}^m$ with $\omega|_P \equiv 0$. The trouble with this result is that as the set of such submanifolds is infinite-dimensional, the theorem is really an infinite-dimensional evolution problem, and so is difficult to solve explicitly.

What we achieve in Definition 3.1 and Theorem 3.2 is to find a special class $\mathcal{C}$ of real analytic $(m-1)$-submanifolds P of $\mathbb{C}^m$ with $\omega|_P \equiv 0$, depending on finitely many real parameters $c_1, \dots, c_n$, such that the evolution equation (2) stays within the class $\mathcal{C}$.

In fact (2) reduces to (1), which is basically the same equation, but is now a first order o.d.e. on $c_1, \dots, c_n$, as functions of t. Thus we have reduced the infinite-dimensional problem of evolving submanifolds in $\mathbb{C}^m$ to a *finite-dimensional* o.d.e., which we may be able to solve explicitly.

4 A geometric interpretation of evolution data with $m = 3$

In [5, Th. 4.9] the author showed that every set of evolution data (P, χ) in $\mathbb{R}^n$ admits a locally transitive symmetry group G in $\mathrm{GL}(n, \mathbb{R})$, and that when $m = 3$ there is a G-invariant surjective map $(\mathbb{R}^n)^* \to \mathfrak{g}$ with kernel 0 or $\mathbb{R}$, where $\mathfrak{g}$ is the Lie algebra of G. Motivated by this, we shall now present a correspondence between sets of linear or affine evolution data (P, χ) with $m = 3$, and symplectic 2-manifolds (Σ, ω) with a transitive Hamiltonian symmetry group.

Let (Σ, ω) be a symplectic 2-manifold, not necessarily compact, and G a connected Lie group with Lie algebra $\mathfrak{g}$ acting faithfully and transitively on Σ. Suppose that G preserves ω and every element of $\mathfrak{g}$ admits a moment map; this is called a *Hamiltonian action*, and holds automatically if Σ is simply-connected. Define V to be the vector space of moment maps of elements of $\mathfrak{g}$, including constant functions.

That is, V is the vector space of smooth maps $f : \Sigma \to \mathbb{R}$ such that $\mathrm{d}f = x \cdot \omega$ for some $x \in \mathfrak{g}$. Then $V \cong \mathbb{R} \oplus \mathfrak{g}$. Define $\psi : \Sigma \to V^*$ by $\psi(x) \cdot f = f(x)$ for all $f \in V$ and $x \in \Sigma$. As G acts transitively on Σ, one can show that ψ is an immersion. Let $P = \psi(\Sigma)$, so that P is an immersed 2-submanifold in V^*.

Now the *Poisson bracket* on (Σ, ω) yields a natural bilinear, antisymmetric map $\{\,,\,\} : V \times V \to V$ given in index notation by $\{f, f'\} = \omega^{ab}(\mathrm{d}f)_a(\mathrm{d}f')_b$, where ω^{ab} is the inverse of ω_{ab}. This makes V into a Lie algebra, which is an extension of the Lie algebra $\mathfrak{g}$ by $\mathbb{R}$.

Thus P is a submanifold in the dual of a Lie algebra. In fact P is a *coadjoint orbit*, that is, an orbit of the coadjoint action on V^* of the connected, simply-connected Lie group associated to V. It is well known that all coadjoint orbits have a natural symplectic structure.

As the Poisson bracket is bilinear and antisymmetric, we can extend it to a linear map $\{\,,\,\} : \Lambda^2 V \to V$. Define $\chi : V^* \to \Lambda^2 V^*$ to be the dual of this linear map. Using the fact that G acts transitively on Σ, it is not difficult to show that $\chi(p)$ is a nonzero element of $\Lambda^2 T_p P \subset \Lambda^2 V^*$ for each $p \in P$. Thus (P, χ) is a set of *linear evolution data* in the vector space V^*.

Actually, it is usually nicer to regard (P, χ) as *affine* evolution data, in the following way. Let $\mathbf{1}$ be the constant function 1 on Σ. Then $\mathbf{1} \in V$. Define $f : V^* \to \mathbb{R}$ by $f(\alpha) = \alpha(\mathbf{1})$. Then $U = f^{-1}(1)$ is a hyperplane in V^*, which contains P. We can regard U as an affine space modelled on $\mathfrak{g}^*$. The restriction of χ to U is an affine map $\chi : U \to \Lambda^2 U$, and (P, χ) is a set of affine evolution data in the affine space U.

We have shown that given a symplectic 2-fold (Σ, ω) with a faithful, transitive, Hamiltonian action of a Lie group G, we can construct sets of linear and affine evolution data with $m = 3$. We now explain how to reverse this construction, so that starting with a set of evolution data with $m = 3$, we construct a symplectic 2-manifold (Σ, ω) and group action. For simplicity we work with the linear case, as affine evolution data in $\mathbb{R}^n$ can always be reduced to linear evolution data in $\mathbb{R}^{n+1}$.

Let (P', χ') be a set of linear evolution data with $m = 3$ in a real vector space W. That is, P' is a connected 2-submanifold of W, lying in no proper vector subspace of W, and $\chi' : W \to \Lambda^2 W$ a linear map such that $\chi'(p) \in \Lambda^2 T_p P' \setminus \{0\}$ for each $p \in P'$. Let ω' be the symplectic structure on P' dual to $\chi'|_{P'}$. Then (P', ω') is a symplectic 2-manifold. We shall define an antisymmetric, bilinear bracket $[\,,\,] : W^* \times W^* \to W^*$ which makes W^* into a Lie algebra.

Regard χ' as an element of $W^* \otimes \Lambda^2 W$, and for each $\alpha, \beta \in W^*$, define $[\alpha, \beta] = \chi' \cdot (\alpha \wedge \beta)$, where '$\cdot$' is the natural pairing between $\Lambda^2 W^*$ and $\Lambda^2 W$. Now W^* is the vector space of linear maps $W \to \mathbb{R}$, so elements of W^* give real functions on P' by restriction. As P' is contained in no proper subspace of W, this map from W^* to functions on P' is injective, so W^* may be viewed as a vector space of functions on P'. Thought of in this way, it is easy to show that the bracket $[\,,\,]$ on W^* is actually the *Poisson bracket* on functions on P', induced by the symplectic structure ω'.

But the Poisson bracket automatically satisfies the Jacobi identity. Thus $[\,,\,]$ makes W^* into a *Lie algebra*. To each element of W^* we associate its Hamiltonian vector field, giving a map $W^* \to \mathrm{Vect}(P')$, where $\mathrm{Vect}(P')$ is the smooth vector fields on P'. This is a *Lie algebra automorphism*, with respect to the usual Lie bracket of vector fields on $\mathrm{Vect}(P')$. Let $\mathfrak{g}$ be the image of W^* in $\mathrm{Vect}(P')$. Then $\mathfrak{g}$ is a finite-dimensional Lie subalgebra of $\mathrm{Vect}(P')$.

It can be shown that either

(a) P' is contained in no affine hyperplane in W and $\mathfrak{g} \cong W^*$, or

(b) P' is contained in an affine hyperplane in W and $\mathfrak{g} \cong W^*/\mathbb{R}$, where $\mathbb{R}$ is an ideal in W^*. Also, (P', χ') reduces to a set of affine evolution data in one fewer dimension.

By Cartan's theorems, there exists a unique connected, simply-connected Lie group G with Lie algebra $\mathfrak{g}$. Choose $p \in P'$, and let $\mathfrak{h}$ be the vector subspace of vector fields in $\mathfrak{g}$ that vanish at p. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, and so corresponds to a unique connected Lie subgroup H of G.

As we can take two functions in W^* to be coordinates near p, the corresponding vectors in T_pP' are linearly independent, and so $\dim \mathfrak{h} = \dim \mathfrak{g} - 2$. Also, there is a natural isomorphism $T_pP' \cong \mathfrak{g}/\mathfrak{h}$. Since the vector fields in $\mathfrak{g}$ are Hamiltonian, they preserve ω'. Thus the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$ preserves the nonzero 2-form ω'_p on $T_pP' \cong \mathfrak{g}/\mathfrak{h}$. As H is connected, it follows that H also preserves ω'_p.

It can be shown that H is closed in G. Then $\Sigma = G/H$ is a connected 2-manifold with a natural G-action. The tangent space $T_H\Sigma$ is isomorphic to $\mathfrak{g}/\mathfrak{h} \cong T_pP'$, and so has a nonzero 2-form ω'_p. As ω'_p is H-invariant, this extends to a G-invariant, nonvanishing 2-form ω on Σ.

This makes Σ into a symplectic 2-manifold with a transitive G-action preserving ω. As G is simply-connected, so is Σ. Thus, every element of $\mathfrak{g}$ admits a moment map for its action on Σ. Therefore by the construction at the beginning of this section, we can associate a set of linear evolution data (P, χ) in a vector space V^* to (Σ, ω) and G.

One can prove that W is naturally isomorphic to V^*, and that this isomorphism identifies P' with an open subset of P, and χ' with χ. The details are left to the reader. To sum up, we have proved the following theorem:

Theorem 4.1 *Let (Σ, ω) be a symplectic 2-manifold, and G a connected Lie group with a faithful, transitive, Hamiltonian action on Σ. Then we can construct sets of linear and affine evolution data (P, χ) with $m = 3$ and $P \cong \Sigma$. Conversely, every set of linear or affine evolution data with $m = 3$ locally arises from this construction.*

All of the quadric examples of [5, §4.1] for $m = 3$ can be easily extracted from this construction. For example:

(i) Let Σ be $\mathcal{S}^2$, and G be SO(3) acting by isometries. Then P is the sphere $x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$.

(ii) Let Σ be the hyperbolic plane $\mathcal{H}^2$, and G be $SO(1,1)_+$ acting by isometries. Then P is half of the hyperboloid $x_1^2 - x_2^2 - x_3^2 = 1$ in $\mathbb{R}^3$.

(iii) Let Σ be $\mathbb{R}^2 \setminus \{0\}$ with the standard volume form, and G be $\mathrm{SL}(2, \mathbb{R})$ acting as usual. Then P is one of the pair of cones $x_1^2 - x_2^2 - x_3^2 = 0$ in $\mathbb{R}^3$.

(iv) Let Σ be $\mathbb{R}^2$ and G be the group of Euclidean transformations $\mathrm{SO}(2) \ltimes \mathbb{R}^2$. Then P is the paraboloid $x_1^2 + x_2^2 + x_3 = 0$ in $\mathbb{R}^3$.

But we can also find interesting new sets of evolution data which are not quadrics. For instance, when we take Σ to be $\mathbb{R}^2$ with its standard volume form, and G to be $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ acting on Σ by affine transformations, we get the following example.

Example 4.2 Consider the map $\psi : \mathbb{R}^2 \to \mathbb{R}^5$ given by

$$\psi : (y_1, y_2) \mapsto \bigl(\tfrac{1}{2}(y_1^2 + y_2^2), \tfrac{1}{2}(y_1^2 - y_2^2), y_1 y_2, y_1, y_2\bigr).$$

The image of ψ is

$$P = \bigl\{(x_1, \ldots, x_5) \in \mathbb{R}^5 : x_1 = \tfrac{1}{2}(x_4^2 + x_5^2),\ x_2 = \tfrac{1}{2}(x_4^2 - x_5^2),\ x_3 = x_4 x_5\bigr\},$$

which is diffeomorphic to $\mathbb{R}^2$. Writing $e_j = \frac{\partial}{\partial x_j}$, calculation shows that

$$\psi_*\bigl(\tfrac{\partial}{\partial y_1}\bigr) = y_1 e_1 + y_1 e_2 + y_2 e_3 + e_4, \quad \psi_*\bigl(\tfrac{\partial}{\partial y_2}\bigr) = y_2 e_1 - y_2 e_2 + y_1 e_3 + e_5,$$

and therefore

$$\begin{aligned} \psi_*\bigl(\tfrac{\partial}{\partial y_1} \wedge \tfrac{\partial}{\partial y_2}\bigr) &= (y_1^2 + y_2^2) e_2 \wedge e_3 + (y_1^2 - y_2^2) e_1 \wedge e_3 - 2 y_1 y_2 e_1 \wedge e_2 \\ &+ y_1(e_1 \wedge e_5 + e_2 \wedge e_5 - e_3 \wedge e_4) + y_2(-e_1 \wedge e_4 + e_2 \wedge e_4 + e_3 \wedge e_5) \\ &+ e_4 \wedge e_5. \end{aligned}$$

Thus if we define an affine map $\chi : \mathbb{R}^5 \to \Lambda^2 \mathbb{R}^5$ by

$$\begin{aligned} &\chi(x_1, \ldots, x_5) = 2x_1 e_2 \wedge e_3 + 2x_2 e_1 \wedge e_3 - 2x_3 e_1 \wedge e_2 \\ &+ x_4(e_1 \wedge e_5 + e_2 \wedge e_5 - e_3 \wedge e_4) + x_5(-e_1 \wedge e_4 + e_2 \wedge e_4 + e_3 \wedge e_5) \\ &+ e_4 \wedge e_5, \end{aligned}$$

then $\chi = \psi_*\bigl(\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}\bigr)$ on P. This implies that (P, χ) is a set of affine evolution data with $m = 3$ and $n = 5$, which does not arise from the construction of [5, §4.1]. So applying Theorem 3.2 will give a family of special Lagrangian 3-folds in $\mathbb{C}^3$. These will be studied at length in §5–§11.

Now if a Lie group G acts transitively on a symplectic 2-manifold Σ then often a Lie subgroup G' of G will also act transitively on Σ, or on some open subset Σ' of Σ. In our next result we consider the relation between the families of SL 3-folds constructed using Σ, G and Σ', G'.

Proposition 4.3 *Let (Σ, ω) be a symplectic 2-manifold, and G a connected Lie group with a faithful, transitive, Hamiltonian action on Σ. Suppose G' is a connected Lie subgroup of G, and Σ' an open orbit of G' in Σ. Then the special Lagrangian 3-folds in $\mathbb{C}^3$ constructed using Σ and G by combining Theorem 4.1 and the method of §3 include all those constructed using Σ' and G'.*

Proof. The construction above gives sets of linear evolution data (P,χ) and $(P'\chi')$ from Σ,G and Σ',G', where P,P' lie in vector spaces $V^*,(V')^*$. It is easy to see that V' is a vector subspace of V, since the Lie algebra $\mathfrak{g}'$ of G' is a vector subspace of $\mathfrak{g}$. Let U be the *annihilator* $(V')^\circ$ of V' in V^*. Then $(V')^*\cong V^*/U$.

Now the construction of SL 3-folds using Σ',G' in §3 involves a 1-parameter family of linear maps $\phi'_t:(V')^*\rightarrow\mathbb{C}^3$ in $\mathcal{C}_{P'}$ satisfying the o.d.e. (1). Let $\phi_t:V^*\rightarrow\mathbb{C}^3$ be the pull-back of ϕ'_t from $(V')^*=V^*/U$ to V^*. It is easy to show that this family ϕ_t also lie in $\mathcal{C}_P$ and satisfy (1), and that the SL 3-fold N' constructed using the ϕ'_t is a subset of the SL 3-fold N constructed using the ϕ_t. Thus the SL 3-folds constructed using Σ',G' are included in those constructed using Σ,G. □

In particular, the family of SL 3-folds coming from Example 4.2, corresponding to the action of $\mathrm{SL}(2,\mathbb{R})\ltimes\mathbb{R}^2$, will include families of SL 3-folds corresponding to subgroups of $\mathrm{SL}(2,\mathbb{R})\ltimes\mathbb{R}^2$. For example, as in part (iii) above we can set $\Sigma'=\mathbb{R}^2\setminus\{0\}$ and $G'=\mathrm{SL}(2,\mathbb{R})$, and as in part (iv) above we can set $\Sigma'=\mathbb{R}^2$ and $G'=\mathrm{SO}(2)\ltimes\mathbb{R}^2$. So the families of SL 3-folds corresponding to parts (iii) and (iv), which we have already considered in [5], will occur as special cases in the family of SL 3-folds to be studied in §5–§11.

Above we set $\Sigma=G/H$, and used the fact that H acts naturally on $T_H\Sigma=\mathfrak{g}/\mathfrak{h}$ preserving ω_H. It is tempting to assume that this action of H on $T_H\Sigma$ is *faithful*, as would be the case in Riemannian rather than symplectic geometry. If this held then H would be a subgroup of $\mathrm{SL}(2,\mathbb{R})$, and would be largest in Example 4.2.

However, H need not act faithfully on $T_H\Sigma$, and in fact G,H can have arbitrarily large dimension. Here is a class of examples in which this happens. Let C be $\mathbb{R}$ or $\mathcal{S}^1$, let K be a connected Lie group acting smoothly, transitively and faithfully on C, and let U be a nonzero vector space of 1-forms on C which is invariant under K.

Define Σ to be T^*C with its canonical symplectic structure ω, and G to be the semidirect product $K\ltimes U$ acting on Σ by

$$(x,y\,\mathrm{d}x)\overset{(\kappa,u)}{\longmapsto}\bigl(\kappa(x),\tfrac{\mathrm{d}\kappa}{\mathrm{d}x}(x)y\,\mathrm{d}x+u(x)\bigr).\tag{3}$$

Here we write a point in Σ as $(x,y\,\mathrm{d}x)$, where x is a coordinate in C with values in $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$, and $y\,\mathrm{d}x$ lies in T^*_xC, so that $y\in\mathbb{R}$. Elements of $K\ltimes U$ are written (κ,u) for $\kappa\in K$ and $u\in U$, so that $\kappa:C\rightarrow C$ is a differentiable map. It is easy to see that (3) defines the action of a Lie group $G=K\ltimes U$ on Σ, which is faithful and transitive and preserves ω.

The possibilities for C and K are

(i) $C=\mathbb{R}$ and $K=\mathbb{R}$ acting by translations.

(ii) $C=\mathbb{R}$ and $K=\mathbb{R}_+\ltimes\mathbb{R}$, acting by $x\overset{(a,b)}{\longmapsto}ax+b$ for $a>0$ and $b\in\mathbb{R}$.

(iii) $C=\mathcal{S}^1$, thought of as $\mathrm{U}(1)$, and $K=\mathrm{U}(1)$ acting by multiplication.

(iv) $C = \mathcal{S}^1$, thought of as $\mathbb{RP}^1$, and $K = \mathrm{PSL}(2,\mathbb{R})$ acting by projective transformations.

In case (iv), one can show that there are no non-zero, finite-dimensional, K-invariant vector spaces of 1-forms U on C, so we rule this case out. In cases (i) and (iii) there are many suitable spaces of 1-forms U, and so we may construct many sets of evolution data (P,χ) in $\mathbb{R}^n$. However, calculation shows that we may always choose coordinates $(x_1,\ldots,x_n)$ on $\mathbb{R}^n$, and so split $\mathbb{R}^n = \mathbb{R}^{n-1}\times\mathbb{R}$, such that $\chi = v\wedge\frac{\partial}{\partial x_n}$, where v is a linear or affine vector field in $\mathbb{R}^{n-1}$, and $P = \gamma\times\mathbb{R}$, where γ is an integral curve of v in $\mathbb{R}^{n-1}$.

Thus, cases (i) and (iii) yield evolution data resulting from combining Examples 4.5 and 4.6 of [5]. The corresponding SL 3-folds will all split as products $\Sigma\times\mathbb{R}$ in $\mathbb{C}^2\times\mathbb{C}$, where Σ is an SL 2-fold in $\mathbb{C}^2$. We are not interested in such examples, so we rule these cases out too.

This leaves case (ii). Here the natural candidates for U are

$$U_k = \bigl\{p(x)\,\mathrm{d}x : p(x) \text{ is a real polynomial of degree } < k\bigr\},$$

for $k\geqslant 1$. In the following example we define the corresponding set of affine evolution data in $\mathbb{R}^{k+2}$, yielded by the construction of Theorem 4.1.

Example 4.4 Choose $k\geqslant 1$, and let $(x_1,\ldots,x_k,y_1,y_2)$ be coordinates on $\mathbb{R}^{k+2}$. Define a map $\psi:\mathbb{R}^2\rightarrow\mathbb{R}^{k+2}$ by $\psi:(x,y)\longmapsto(x,x^2,\ldots,x^k,y,xy)$. The image of ψ is

$$P = \bigl\{(x_1,\ldots,x_k,y_1,y_2)\in\mathbb{R}^{k+2} : x_j = (x_1)^j \text{ for } j=2,\ldots,k,\; y_2 = y_1x_1\bigr\},$$

which is diffeomorphic to $\mathbb{R}^2$. Calculation shows that

$$\psi_*\bigl(\tfrac{\partial}{\partial x}\bigr) = y\tfrac{\partial}{\partial y_2} + \tfrac{\partial}{\partial x_1} + 2x\tfrac{\partial}{\partial x_2} + \cdots + kx^{k-1}\tfrac{\partial}{\partial x_k} \quad\text{and}\quad \psi_*\bigl(\tfrac{\partial}{\partial y}\bigr) = \tfrac{\partial}{\partial y_1} + x\tfrac{\partial}{\partial y_2},$$

and therefore

$$\begin{aligned}\psi_*\bigl(\tfrac{\partial}{\partial x}\wedge\tfrac{\partial}{\partial y}\bigr) &= -y\tfrac{\partial}{\partial y_1}\wedge\tfrac{\partial}{\partial y_2} + \tfrac{\partial}{\partial x_1}\wedge\tfrac{\partial}{\partial y_1} + 2x\tfrac{\partial}{\partial x_2}\wedge\tfrac{\partial}{\partial y_1} + \cdots + kx^{k-1}\tfrac{\partial}{\partial x_k}\wedge\tfrac{\partial}{\partial y_1}\\ &\quad + x\tfrac{\partial}{\partial x_1}\wedge\tfrac{\partial}{\partial y_2} + 2x^2\tfrac{\partial}{\partial x_2}\wedge\tfrac{\partial}{\partial y_2} + \cdots + kx^k\tfrac{\partial}{\partial x_k}\wedge\tfrac{\partial}{\partial y_2}\\ &= -y_1\tfrac{\partial}{\partial y_1}\wedge\tfrac{\partial}{\partial y_2} + \tfrac{\partial}{\partial x_1}\wedge\tfrac{\partial}{\partial y_1} + 2x_1\tfrac{\partial}{\partial x_2}\wedge\tfrac{\partial}{\partial y_1} + \cdots + kx_{k-1}\tfrac{\partial}{\partial x_k}\wedge\tfrac{\partial}{\partial y_1}\\ &\quad + x_1\tfrac{\partial}{\partial x_1}\wedge\tfrac{\partial}{\partial y_2} + 2x_2\tfrac{\partial}{\partial x_2}\wedge\tfrac{\partial}{\partial y_2} + \cdots + kx_k\tfrac{\partial}{\partial x_k}\wedge\tfrac{\partial}{\partial y_2}.\end{aligned}$$

Thus if we define an affine map $\chi:\mathbb{R}^{k+2}\rightarrow\Lambda^2\mathbb{R}^{k+2}$ by

$$\begin{aligned}&\chi(x_1,\ldots,x_k,y_1,y_2)\\ &\quad = -2y_1\tfrac{\partial}{\partial y_1}\wedge\tfrac{\partial}{\partial y_2} + 2\tfrac{\partial}{\partial x_1}\wedge\tfrac{\partial}{\partial y_1} + 4x_1\tfrac{\partial}{\partial x_2}\wedge\tfrac{\partial}{\partial y_1} + \cdots + 2kx_{k-1}\tfrac{\partial}{\partial x_k}\wedge\tfrac{\partial}{\partial y_1}\\ &\qquad + 2x_1\tfrac{\partial}{\partial x_1}\wedge\tfrac{\partial}{\partial y_2} + 4x_2\tfrac{\partial}{\partial x_2}\wedge\tfrac{\partial}{\partial y_2} + \cdots + 2kx_k\tfrac{\partial}{\partial x_k}\wedge\tfrac{\partial}{\partial y_2},\end{aligned}$$

then $\chi = 2\psi_*\bigl(\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\bigr)$ on P. This implies that (P,χ) is a set of affine evolution data with $m=3$ and $n=k+2$. So applying Theorem 3.2 gives a family of special Lagrangian 3-folds in $\mathbb{C}^3$, which will be studied in §12.

5 A construction of SL 3-folds in $\mathbb{C}^3$

We now apply the 'evolution equation' construction of §3 to the set of affine evolution data defined in Example 4.2. As in Example 4.2, let P be the image in $\mathbb{R}^5$ of the map $\psi : \mathbb{R}^2 \to \mathbb{R}^5$ given by

$$\psi : (y_1, y_2) \mapsto \bigl(\tfrac{1}{2}(y_1^2 + y_2^2), \tfrac{1}{2}(y_1^2 - y_2^2), y_1 y_2, y_1, y_2\bigr), \tag{4}$$

and define $\chi : \mathbb{R}^5 \to \Lambda^2\mathbb{R}^5$ by

$$\begin{aligned} &\chi(x_1, \ldots, x_5) = 2x_1 e_2 \wedge e_3 + 2x_2 e_1 \wedge e_3 - 2x_3 e_1 \wedge e_2 \\ &+ x_4(e_1 \wedge e_5 + e_2 \wedge e_5 - e_3 \wedge e_4) + x_5(-e_1 \wedge e_4 + e_2 \wedge e_4 + e_3 \wedge e_5) \\ &+ e_4 \wedge e_5, \end{aligned} \tag{5}$$

where $e_j = \frac{\partial}{\partial x_j}$. Then (P, χ) is a set of affine evolution data.

Let $\mathbf{z}_1, \ldots, \mathbf{z}_6$ be vectors in $\mathbb{C}^3$, and define an affine map $\phi : \mathbb{R}^5 \to \mathbb{C}^3$ by

$$\phi : (x_1, \ldots, x_5) \mapsto x_1 \mathbf{z}_1 + \cdots + x_5 \mathbf{z}_5 + \mathbf{z}_6. \tag{6}$$

Then as $\phi_*(e_k) = \mathbf{z}_k$ for $k \leqslant 5$, from (5) we see that

$$\begin{aligned} \phi^*(\omega) \cdot \chi = {} & 2x_1 \omega(\mathbf{z}_2, \mathbf{z}_3) + 2x_2 \omega(\mathbf{z}_1, \mathbf{z}_3) - 2x_3 \omega(\mathbf{z}_1, \mathbf{z}_2) \\ & + x_4\bigl(\omega(\mathbf{z}_1, \mathbf{z}_5) + \omega(\mathbf{z}_2, \mathbf{z}_5) - \omega(\mathbf{z}_3, \mathbf{z}_4)\bigr) \\ & + x_5\bigl(-\omega(\mathbf{z}_1, \mathbf{z}_4) + \omega(\mathbf{z}_2, \mathbf{z}_4) + \omega(\mathbf{z}_3, \mathbf{z}_5)\bigr) + \omega(\mathbf{z}_4, \mathbf{z}_5). \end{aligned}$$

Thus $\phi^*(\omega)|_P \equiv 0$ if and only if

$$\omega(\mathbf{z}_2, \mathbf{z}_3) = \omega(\mathbf{z}_1, \mathbf{z}_3) = \omega(\mathbf{z}_1, \mathbf{z}_2) = 0, \tag{7}$$

$$\omega(\mathbf{z}_1, \mathbf{z}_5) + \omega(\mathbf{z}_2, \mathbf{z}_5) - \omega(\mathbf{z}_3, \mathbf{z}_4) = 0, \tag{8}$$

$$-\omega(\mathbf{z}_1, \mathbf{z}_4) + \omega(\mathbf{z}_2, \mathbf{z}_4) + \omega(\mathbf{z}_3, \mathbf{z}_5) = 0, \tag{9}$$

$$\text{and} \qquad \omega(\mathbf{z}_4, \mathbf{z}_5) = 0. \tag{10}$$

Now ϕ lies in the set $\mathcal{C}_P$ of Definition 3.1 if and only if equations (7)–(10) hold and $\phi(P)$ is 2-dimensional, which is an open condition on ϕ. Hence the $\mathbf{z}_j$ have 36 real parameters satisfying 6 real equations, so that $\mathcal{C}_P$ has dimension 30.

Motivated by (1), define a 'cross product' $\times : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$ by

$$(\mathbf{r} \times \mathbf{s})^b = \mathbf{r}^{a_1} \mathbf{s}^{a_2} (\mathrm{Re}\,\Omega)_{a_1 a_2 a_3} g^{a_3 b}, \tag{11}$$

regarding $\mathbb{C}^3$ as a real vector space, and using the index notation for tensors on $\mathbb{C}^3$. Calculation shows that in complex coordinates, we have

$$(r_1, r_2, r_3) \times (s_1, s_2, s_3) = \tfrac{1}{2}(\bar{r}_2 \bar{s}_3 - \bar{r}_3 \bar{s}_2, \bar{r}_3 \bar{s}_1 - \bar{r}_1 \bar{s}_3, \bar{r}_1 \bar{s}_2 - \bar{r}_2 \bar{s}_1), \tag{12}$$

so that '$\times$' is complex anti-bilinear. Note that this cross product is equivariant under the action of SU(3) on $\mathbb{C}^3$.

In §3 we explained how to construct special Lagrangian m-folds using an evolution equation (1) for $\phi \in \mathcal{C}_P$. We shall write this equation out explicitly for ϕ of the form (6). Let $\mathbf{z}_1(t), \ldots, \mathbf{z}_6(t)$ be differentiable functions $\mathbb{R} \to \mathbb{C}^3$, and define ϕ_t by (6) for $t \in \mathbb{R}$. Then, comparing equations (1), (5) and (11), we see that (1) holds for the family $\{\phi_t : t \in \mathbb{R}\}$ if and only if

$$\begin{aligned}\frac{\mathrm{d}\phi_t}{\mathrm{d}t}(x_1, \ldots, x_5) &= 2x_1 \mathbf{z}_2 \times \mathbf{z}_3 + 2x_2 \mathbf{z}_1 \times \mathbf{z}_3 - 2x_3 \mathbf{z}_1 \times \mathbf{z}_2 \\ &+ x_4(\mathbf{z}_1 \times \mathbf{z}_5 + \mathbf{z}_2 \times \mathbf{z}_5 - \mathbf{z}_3 \times \mathbf{z}_4) + x_5(-\mathbf{z}_1 \times \mathbf{z}_4 + \mathbf{z}_2 \times \mathbf{z}_4 + \mathbf{z}_3 \times \mathbf{z}_5) + \mathbf{z}_4 \times \mathbf{z}_5.\end{aligned}$$

Using (6) we get expressions for $\mathrm{d}\mathbf{z}_j/\mathrm{d}t$ for $j = 1, \ldots, 6$. So applying Theorem 3.2, we prove:

Theorem 5.1 *Suppose $\mathbf{z}_1, \ldots, \mathbf{z}_6 : \mathbb{R} \to \mathbb{C}^3$ are differentiable functions satisfying equations* (7)–(10) *at $t = 0$ and*

$$\frac{\mathrm{d}\mathbf{z}_1}{\mathrm{d}t} = 2\mathbf{z}_2 \times \mathbf{z}_3, \qquad \frac{\mathrm{d}\mathbf{z}_2}{\mathrm{d}t} = 2\mathbf{z}_1 \times \mathbf{z}_3, \qquad \frac{\mathrm{d}\mathbf{z}_3}{\mathrm{d}t} = -2\mathbf{z}_1 \times \mathbf{z}_2, \tag{13}$$

$$\frac{\mathrm{d}\mathbf{z}_4}{\mathrm{d}t} = \mathbf{z}_1 \times \mathbf{z}_5 + \mathbf{z}_2 \times \mathbf{z}_5 - \mathbf{z}_3 \times \mathbf{z}_4, \quad \frac{\mathrm{d}\mathbf{z}_5}{\mathrm{d}t} = -\mathbf{z}_1 \times \mathbf{z}_4 + \mathbf{z}_2 \times \mathbf{z}_4 + \mathbf{z}_3 \times \mathbf{z}_5, \tag{14}$$

$$\text{and} \qquad \frac{\mathrm{d}\mathbf{z}_6}{\mathrm{d}t} = \mathbf{z}_4 \times \mathbf{z}_5 \tag{15}$$

for all $t \in \mathbb{R}$, where '$\times$' is as in (12)*. Define a subset N of $\mathbb{C}^3$ by*

$$\begin{aligned} N = \bigl\{ &\tfrac{1}{2}(y_1^2 + y_2^2)\, \mathbf{z}_1(t) + \tfrac{1}{2}(y_1^2 - y_2^2)\, \mathbf{z}_2(t) + y_1 y_2\, \mathbf{z}_3(t) \\ &+ y_1\, \mathbf{z}_4(t) + y_2\, \mathbf{z}_5(t) + \mathbf{z}_6(t) : y_1, y_2, t \in \mathbb{R} \bigr\}. \end{aligned} \tag{16}$$

Then N is a special Lagrangian 3*-fold in $\mathbb{C}^3$ wherever it is nonsingular.*

The results of [5, §3] also show that if (7)–(10) hold at $t = 0$ then they hold for all $t \in \mathbb{R}$, and that given initial values $\mathbf{z}_1(0), \ldots, \mathbf{z}_6(0)$, there exist unique solutions $\mathbf{z}_1(t), \ldots, \mathbf{z}_6(t)$ to (13)–(15) for t in $(-\epsilon, \epsilon)$ and some small $\epsilon > 0$. In fact it will follow from later results that solutions always exist for all $t \in \mathbb{R}$, and this is why we have used $t \in \mathbb{R}$ rather than $t \in (-\epsilon, \epsilon)$ above.

5.1 Transformation of $\mathbf{z}_1, \ldots, \mathbf{z}_6$ under $\mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$

The evolution data (P, χ) we used above was derived in §4 from the action of $\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ on $\mathbb{R}^2$ by symplectic affine transformations. We shall now show that the construction of Theorem 5.1 is invariant under the action not just of $\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$, but under the full group $\mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ of affine transformations of $\mathbb{R}^2$. That is, we shall define an action of $\mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ on the set of solutions $\mathbf{z}_1, \ldots, \mathbf{z}_6$ of (13)–(15) which fixes the corresponding SL 3-folds N of (16).

Consider the affine transformation of $\mathbb{R}^2$ given by

$$(y_1, y_2) \mapsto (ay_1 + by_2 + e, cy_1 + dy_2 + f), \tag{17}$$

where $a, b, c, d, e, f \in \mathbb{R}$, and the determinant $\delta = ad - bc$ is nonzero. Suppose that $\mathbf{z}_1, \dots, \mathbf{z}_6 : \mathbb{R} \to \mathbb{C}^3$ satisfy (13)–(15). The natural way to make the transformation (17) act upon $\mathbf{z}_1, \dots, \mathbf{z}_6$ is to define $\mathbf{z}'_1, \dots, \mathbf{z}'_6$ by equating coefficients of polynomials in y'_1, y'_2 in the equation

$$\begin{aligned}&\tfrac{1}{2}(y_1^2 + y_2^2)\,\mathbf{z}_1 + \tfrac{1}{2}(y_1^2 - y_2^2)\,\mathbf{z}_2 + y_1 y_2\,\mathbf{z}_3 + y_1\,\mathbf{z}_4 + y_2\,\mathbf{z}_5 + \mathbf{z}_6 = \\ &\tfrac{1}{2}\big((y'_1)^2 + (y'_2)^2\big)\mathbf{z}'_1 + \tfrac{1}{2}((y'_1)^2 - (y'_2)^2\big)\mathbf{z}'_2 + y'_1 y'_2 \mathbf{z}'_3 + y'_1 \mathbf{z}'_4 + y'_2 \mathbf{z}'_5 + \mathbf{z}'_6,\end{aligned}$$

where $y_1 = a y'_1 + b y'_2 + e$ and $y_2 = c y'_1 + d y'_2 + f$.

Here each side is a polynomial in y'_1, y'_2 with values in $\mathbb{C}^3$, taken from (16). A straightforward calculation gives expressions for $\mathbf{z}'_1, \dots, \mathbf{z}'_6$ in terms of $\mathbf{z}_1, \dots, \mathbf{z}_6$ and $a, \dots, f$, so that for example

$$\mathbf{z}'_1(t) = \tfrac{1}{2}(a^2 + b^2 + c^2 + d^2)\mathbf{z}_1(t) + \tfrac{1}{2}(a^2 + b^2 - c^2 - d^2)\mathbf{z}_2(t) + (ac + bd)\mathbf{z}_3(t).$$

If the transformation (17) lies in $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ then it is easy to see from the construction of the evolution data out of the action of $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ that $\mathbf{z}'_1, \dots, \mathbf{z}'_6$ must satisfy (13)–(15), and yield exactly the same SL 3-fold N in (16) as $\mathbf{z}_1, \dots, \mathbf{z}_6$ do.

However, if (17) lies in $\mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ rather than $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ then the $\mathbf{z}'_j$ will not in general satisfy (13)–(15). This is because the data χ of (5) is essentially the same as $\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}$, but (17) multiplies $\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}$ by $\delta = ad - bc$. Thus, in (13)–(15) the $\mathrm{d}\mathbf{z}_j/\mathrm{d}t$ are also multiplied by δ. We deal with this by replacing t by $t' = \delta^{-1} t$, and then the $\mathbf{z}'_j$ satisfy (13)–(15) with respect to the new time variable t'. Hence we prove:

Proposition 5.2 *Suppose that* $\mathbf{z}_1, \dots, \mathbf{z}_6 : \mathbb{R} \to \mathbb{C}^3$ *satisfy* (13)–(15). *Let* $a, b, c, d, e, f \in \mathbb{R}$ *with* $\delta = ad - bc \neq 0$, *and define* $\mathbf{z}'_1, \dots, \mathbf{z}'_6 : \mathbb{R} \to \mathbb{C}^3$ *by*

$$\begin{aligned}\mathbf{z}'_1(t) = {}& \tfrac{1}{2}(a^2 + b^2 + c^2 + d^2)\mathbf{z}_1(\delta t) + \tfrac{1}{2}(a^2 + b^2 - c^2 - d^2)\mathbf{z}_2(\delta t) \\ &+ (ac + bd)\mathbf{z}_3(\delta t),\end{aligned} \tag{18}$$

$$\begin{aligned}\mathbf{z}'_2(t) = {}& \tfrac{1}{2}(a^2 - b^2 + c^2 - d^2)\mathbf{z}_1(\delta t) + \tfrac{1}{2}(a^2 - b^2 - c^2 + d^2)\mathbf{z}_2(\delta t) \\ &+ (ac - bd)\mathbf{z}_3(\delta t),\end{aligned} \tag{19}$$

$$\mathbf{z}'_3(t) = (ab + cd)\mathbf{z}_1(\delta t) + (ab - cd)\mathbf{z}_2(\delta t) + (ad + bc)\mathbf{z}_3(\delta t), \tag{20}$$

$$\begin{aligned}\mathbf{z}'_4(t) = {}& (ae + cf)\mathbf{z}_1(\delta t) + (ae - cf)\mathbf{z}_2(\delta t) + (af + ce)\mathbf{z}_3(\delta t) \\ &+ a\,\mathbf{z}_4(\delta t) + c\,\mathbf{z}_5(\delta t),\end{aligned} \tag{21}$$

$$\begin{aligned}\mathbf{z}'_5(t) = {}& (be + df)\mathbf{z}_1(\delta t) + (be - df)\mathbf{z}_2(\delta t) + (bf + de)\mathbf{z}_3(\delta t) \\ &+ b\,\mathbf{z}_4(\delta t) + d\,\mathbf{z}_5(\delta t),\end{aligned} \tag{22}$$

$$\begin{aligned}\mathbf{z}'_6(t) = {}& \tfrac{1}{2}(e^2 + f^2)\mathbf{z}_1(\delta t) + \tfrac{1}{2}(e^2 - f^2)\mathbf{z}_2(\delta t) + ef\,\mathbf{z}_3(\delta t) \\ &+ e\,\mathbf{z}_4(\delta t) + f\,\mathbf{z}_5(\delta t) + \mathbf{z}_6(\delta t).\end{aligned} \tag{23}$$

Then $\mathbf{z}'_1, \dots, \mathbf{z}'_6$ *satisfy* (13)–(15). *Furthermore, the* $\mathbf{z}'_j$ *satisfy* (7)–(10) *if and only if the* $\mathbf{z}_j$ *do, and in this case the special Lagrangian* 3*-folds* N, N' *constructed in* (16) *from the* $\mathbf{z}_j$ *and* $\mathbf{z}'_j$ *are the same.*

Suppose we are given solutions $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 : \mathbb{R} \to \mathbb{C}^3$ to (13), and we wish to solve (14) for $\mathbf{z}_4$ and $\mathbf{z}_5$. Now (14) is *linear* in $\mathbf{z}_4, \mathbf{z}_5$, so one solution is $\mathbf{z}_4 = \mathbf{z}_5 = 0$. Apply the proposition with $a = d = 1$ and $b = c = 0$ and arbitrary values of e, f. It gives new solutions $\mathbf{z}'_1, \ldots, \mathbf{z}'_5$ to (13) and (14), where

$$\mathbf{z}'_1 = \mathbf{z}_1, \;\; \mathbf{z}'_2 = \mathbf{z}_2, \;\; \mathbf{z}'_3 = \mathbf{z}_3, \;\; \mathbf{z}'_4 = e\,\mathbf{z}_1 + e\,\mathbf{z}_2 + f\,\mathbf{z}_3 \;\text{ and }\; \mathbf{z}'_5 = f\,\mathbf{z}_1 - f\,\mathbf{z}_2 + e\,\mathbf{z}_3.$$

This gives:

Corollary 5.3 *Suppose* $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 : \mathbb{R} \to \mathbb{C}^3$ *satisfy* (13)*. Define*

$$\mathbf{z}_4 = e\,\mathbf{z}_1 + e\,\mathbf{z}_2 + f\,\mathbf{z}_3 \quad \textit{and} \quad \mathbf{z}_5 = f\,\mathbf{z}_1 - f\,\mathbf{z}_2 + e\,\mathbf{z}_3,$$

for $e, f \in \mathbb{R}$*. Then* $\mathbf{z}_4, \mathbf{z}_5$ *satisfy* (14)*.*

This will be helpful later in solving (14), given solutions to (13).

5.2 Discussion of the construction

Here is a parameter count for family of the special Lagrangian 3-folds N in $\mathbb{C}^3$ constructed by the theorem. The initial data $\mathbf{z}_1(0), \ldots, \mathbf{z}_6(0)$ has 36 real parameters, as each $\mathbf{z}_j(0)$ lies in $\mathbb{C}^3$. These are subject to 6 real conditions (7)–(10), reducing them to 30 real parameters. That is, $\dim \mathcal{C}_P = 30$ in the notation of Definition 3.1, so the family of curves in $\mathcal{C}_P$ has dimension 29.

However, we saw in §5.1 that $\mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ acts on this family of curves in $\mathcal{C}_P$, and two curves related by the group action give the same 3-fold. As $\mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ has dimension 6, this means that the family of distinct SL 3-folds in $\mathbb{C}^3$ constructed above has dimension $29 - 6 = 23$.

If we identify SL 3-folds isomorphic under automorphisms of $\mathbb{C}^3$, the dimension reduces still further. The appropriate automorphism group is $\mathrm{SU}(3) \ltimes \mathbb{C}^3$, with dimension 14. Thus the family of distinct SL 3-folds in $\mathbb{C}^3$ up to automorphisms of $\mathbb{C}^3$ has dimension $23 - 14 = 9$.

So the number of interesting real parameters in the construction of Theorem 5.1 is 9. For comparison, the number of interesting parameters in the construction of [5, §6] is 3, and the number in the construction of [5, §7] with $m = 3$ is 2. Hence the construction above is quite a lot more general than those of [5, §6], and [5, §7] with $m = 3$. In fact the $m = 3$ cases of [5, §7], discussed in [5, Ex. 7.4 & Ex. 7.5], occur as special cases of the construction above.

In §6–§11 we will study the solutions of the o.d.e.s (13)–(15), and so construct special Lagrangian 3-folds in $\mathbb{C}^3$. The way we have divided the equations up suggests a three-stage solution process. For (13) shows that $\mathrm{d}\mathbf{z}_1/\mathrm{d}t$, $\mathrm{d}\mathbf{z}_2/\mathrm{d}t$ and $\mathrm{d}\mathbf{z}_3/\mathrm{d}t$ depend only on $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$, and not on $\mathbf{z}_4, \mathbf{z}_5$ or $\mathbf{z}_6$. Thus in the first stage we solve the nonlinear equations (13) for $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$, ignoring $\mathbf{z}_4, \mathbf{z}_5$ and $\mathbf{z}_6$.

Then in the second stage we regard $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ as fixed, and solve equations (14) for $\mathbf{z}_4$ and $\mathbf{z}_5$. Notice that (14) are *linear* in $\mathbf{z}_4, \mathbf{z}_5$, which makes them much easier to solve. Also, Corollary 5.3 gives us two of the six solutions automatically.

Finally, in the third stage we regard $\mathbf{z}_1, \ldots, \mathbf{z}_5$ as fixed and solve (15) for $\mathbf{z}_6$, which is just a matter of integration.

Now the first stage reduces to a problem we have already studied in [5]. By ignoring $\mathbf{z}_4, \mathbf{z}_5$ and $\mathbf{z}_6$, we are effectively considering maps $\phi : \mathbb{R}^3 \to \mathbb{C}^3$ given by $\phi : (x_1, x_2, x_3) \mapsto x_1\mathbf{z}_1 + x_2\mathbf{z}_2 + x_3\mathbf{z}_3$. Then P in $\mathbb{R}^3$ is the set of (x_1, x_2, x_3) of the form $\bigl(\frac{1}{2}(y_1^2+y_2^2), \frac{1}{2}(y_1^2-y_2^2), y_1y_2\bigr)$ for $y_1, y_2 \in \mathbb{R}$. This satisfies $x_1^2 = x_2^2+x_3^2$.

Thus, we are evolving the image of a quadric cone in $\mathbb{R}^3$ under a linear map $\mathbb{R}^3 \to \mathbb{C}^3$. This is exactly the what we did in [5, §6]. The equations (13) are in fact equivalent to the problem considered in [5, §6], so we shall use the material of [5, §6] to understand their solutions.

Here is some notation that will be useful. Define a map $\Phi : \mathbb{R}^3 \to \mathbb{C}^3$ by

$$\begin{aligned}\Phi(y_1, y_2, t) = \tfrac{1}{2}(y_1^2 + y_2^2)\,\mathbf{z}_1(t) + \tfrac{1}{2}(y_1^2 - y_2^2)\,\mathbf{z}_2(t) + y_1y_2\,\mathbf{z}_3(t) \\ + y_1\,\mathbf{z}_4(t) + y_2\,\mathbf{z}_5(t) + \mathbf{z}_6(t).\end{aligned} \tag{24}$$

Then the SL 3-fold N of (16) is the image of Φ, that is, $N = \bigl\{\Phi(y_1, y_2, t) : y_1, y_2, t \in \mathbb{R}\bigr\}$.

If Φ is an immersion then N is a nonsingular immersed 3-submanifold. The points where Φ is not an immersion generally lead to *singularities* of N. We will study the points where Φ is not an immersion in §6. In particular, we will show that Φ is an immersion outside a set of real codimension one in the family of all Φ generated in Theorem 5.1. Thus, *generic* SL 3-folds N from Theorem 5.1 are nonsingular as immersed 3-submanifolds.

Another question we shall be interested in is whether the maps Φ are *periodic* in t. That is, we wish to know whether there exists $T > 0$ such that $\Phi(y_1, y_2, t+T) = \Phi(y_1, y_2, t)$ for all $y_1, y_2, t \in \mathbb{R}$. If this holds then we can regard Φ as mapping $\mathbb{R}^2 \times \mathcal{S}^1 \to \mathbb{C}^3$ rather than $\mathbb{R}^3 \to \mathbb{C}^3$, where $\mathcal{S}^1 = \mathbb{R}/T\mathbb{Z}$, so that if Φ is an immersion then N is an immersed copy of $\mathbb{R}^2 \times \mathcal{S}^1$ rather than $\mathbb{R}^3$.

Periodic solutions are interesting they give us examples of SL 3-folds in $\mathbb{C}^3$ with different topologies, and because they are often suitable local models for singularities of SL 3-folds in Calabi–Yau 3-folds, whereas the non-periodic solutions usually are not suitable because they are not closed in $\mathbb{C}^3$, or for other reasons.

6 Singularities of these SL 3-folds

We shall now study the singularities of the special Lagrangian 3-folds constructed in Theorem 5.1. A good way to do this is to use the map $\Phi : \mathbb{R}^3 \to \mathbb{C}^3$ defined in (24). Clearly Φ is smooth. If at each $(y_1, y_2, t) \in \mathbb{R}^3$ its derivative $\mathrm{d}\Phi|_{(y_1,y_2,t)} : \mathbb{R}^3 \to \mathbb{C}^3$ is injective then Φ is an *immersion*, and $N = \operatorname{Image}\Phi$ is nonsingular as an immersed 3-submanifold.

Thus the singularities of N come from points (y_1, y_2, t) for which $\mathrm{d}\Phi|_{(y_1,y_2,t)}$ is not injective. Generically, if $\mathrm{d}\Phi|_{(y_1,y_2,t)}$ is not injective then N is singular at $\Phi(y_1, y_2, t)$. But we will see in cases (a) and (b) of §8.3 that it can happen that Φ is not an immersion, but N is a subset of a nonsingular 3-fold, so that the apparent singularity is due to badly chosen coordinates.

We begin with a couple of lemmas about Φ. The first is true by construction.

Lemma 6.1 *The map Φ of* (24) *satisfies*

$$\omega\Big(\frac{\partial\Phi}{\partial y_1},\frac{\partial\Phi}{\partial y_2}\Big)=\omega\Big(\frac{\partial\Phi}{\partial y_1},\frac{\partial\Phi}{\partial t}\Big)=\omega\Big(\frac{\partial\Phi}{\partial y_2},\frac{\partial\Phi}{\partial t}\Big)=0 \\ \text{and}\qquad \frac{\partial\Phi}{\partial y_1}\times\frac{\partial\Phi}{\partial y_2}=\frac{\partial\Phi}{\partial t}, \tag{25}$$

where '$\times$' is defined in (12).

The second gives a simple criterion to decide whether Φ is an immersion.

Lemma 6.2 *The map Φ of* (24) *is an immersion near $(y_1,y_2,t)\in\mathbb{R}^3$ if and only if $\frac{\partial\Phi}{\partial t}(y_1,y_2,t)\neq 0$, or equivalently if and only if $\frac{\partial\Phi}{\partial y_1}(y_1,y_2,t)$ and $\frac{\partial\Phi}{\partial y_2}(y_1,y_2,t)$ are linearly independent.*

Proof. Since $\omega\big(\frac{\partial\Phi}{\partial y_1},\frac{\partial\Phi}{\partial y_2}\big)=0$ by (25), one can show from $\frac{\partial\Phi}{\partial y_1}\times\frac{\partial\Phi}{\partial y_2}=\frac{\partial\Phi}{\partial t}$ that $\frac{\partial\Phi}{\partial y_1}$, $\frac{\partial\Phi}{\partial y_2}$ and $\frac{\partial\Phi}{\partial t}$ are linearly independent if and only if $\frac{\partial\Phi}{\partial t}\neq 0$, or equivalently if and only if $\frac{\partial\Phi}{\partial y_1}$ and $\frac{\partial\Phi}{\partial y_2}$ are linearly independent. But Φ is an immersion near (y_1,y_2,t) if and only if $\frac{\partial\Phi}{\partial y_1}$, $\frac{\partial\Phi}{\partial y_2}$ and $\frac{\partial\Phi}{\partial t}$ are linearly independent. $\square$

As in §5.2, we can do a parameter count on the family of *singular* SL 3-folds in $\mathbb{C}^3$ constructed above. Consider first the condition that Φ should not be an immersion at (0,0,0). By Lemma 6.2, this happens if and only if $\frac{\partial\Phi}{\partial y_1}(0,0,0)$ and $\frac{\partial\Phi}{\partial y_2}(0,0,0)$ are linearly dependent, that is, if $\mathbf{z}_4(0)$ and $\mathbf{z}_5(0)$ are linearly dependent.

The set of linearly dependent pairs $\mathbf{z}_4(0),\mathbf{z}_5(0)$ in $\mathbb{C}^3$ has dimension 7. Thus the set of initial data $\mathbf{z}_1(0),\ldots,\mathbf{z}_6(0)$ with $\mathbf{z}_4(0)$ and $\mathbf{z}_5(0)$ linearly dependent has $24+7=31$ real parameters. These are subject to 6 real conditions (7)–(10). But one of these, $\omega(\mathbf{z}_4,\mathbf{z}_5)=0$, holds automatically as $\mathbf{z}_4(0)$ and $\mathbf{z}_5(0)$ are linearly dependent.

So the set of initial data $\mathbf{z}_1(0),\ldots,\mathbf{z}_6(0)$ satisfying (7)–(10) with $\mathbf{z}_4(0)$ and $\mathbf{z}_5(0)$ linearly dependent has $31-5=26$ real parameters. For comparison, we saw in §5.2 that the set of initial data $\mathbf{z}_1(0),\ldots,\mathbf{z}_6(0)$ satisfying (7)–(10) has 30 real parameters. Therefore the condition that Φ should not be an immersion at $(0,0,0)$ is of real codimension 4. By symmetry, the condition for Φ not to be an immersion at any given point in $\mathbb{R}^3$ is also of codimension 4.

Thus we expect the family of singular 3-folds N arising from Theorem 5.1 to be of codimension $4-3=1$ in the family of all such 3-folds. So the family of distinct singular SL 3-folds in $\mathbb{C}^3$ from Theorem 5.1, up to automorphisms of $\mathbb{C}^3$, should have dimension 8. In particular, for generic 3-folds N arising from Theorem 5.1, Φ is an immersion, and N is a nonsingular immersed 3-submanifold.

Next we will describe the singularities of the 3-folds N of Theorem 5.1 fairly explicitly, by modelling N near a singular point. A good way to do this is to

expand Φ as a power series about the singular point, to low order. For simplicity, we take the singular point to be at (0,0,0) in $\mathbb{R}^3$ and $\mathbb{C}^3$.

So let $\mathbf{z}_1,\ldots,\mathbf{z}_6$ and Φ satisfy all the conditions of §5, and suppose that $\Phi(0,0,0)=\mathbf{z}_6(0)=0$ and that $\mathrm{d}\Phi|_{(0,0,0)}$ is not injective. As above, this holds if and only if $\mathbf{z}_4(0)$ and $\mathbf{z}_5(0)$ are linearly dependent. We will expand Φ as a power series about 0 up to second order, and use this to describe N near its singular point 0.

Now in §5.1 we described an action of $\mathrm{GL}(2,\mathbb{R})\ltimes\mathbb{R}^2$ on the set of maps Φ satisfying the conditions of §5, that acts trivially on the corresponding SL 3-folds N. Since we are really interested in N rather than Φ, we shall use this action to put Φ in a more convenient form. Under the rotation $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)=\left(\begin{smallmatrix} \cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right)$, with $e=f=0$, we see from (21) and (22) that $\mathbf{z}_4(0)$ and $\mathbf{z}_5(0)$ are transformed to

$$\mathbf{z}_4'(0)=\cos\theta\,\mathbf{z}_4(0)-\sin\theta\,\mathbf{z}_5(0)\quad\text{and}\quad\mathbf{z}_5'(0)=\sin\theta\,\mathbf{z}_4(0)+\cos\theta\,\mathbf{z}_5(0).$$

As $\mathbf{z}_4(0)$ and $\mathbf{z}_5(0)$ are linearly dependent, we may choose θ so that $\mathbf{z}_5'(0)=0$.

So suppose that $\mathbf{z}_5(0)=0$. Take the initial data to be

$$\mathbf{z}_1(0)=\mathbf{v}+\mathbf{w},\;\;\mathbf{z}_2(0)=\mathbf{v}-\mathbf{w},\;\;\mathbf{z}_3(0)=\mathbf{x},\;\;\mathbf{z}_4(0)=\mathbf{u},\;\;\mathbf{z}_5(0)=\mathbf{z}_6(0)=0,$$

for vectors $\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{x}$ in $\mathbb{C}^3$. Equations (7)–(10) then reduce to

$$\omega(\mathbf{u},\mathbf{w})=\omega(\mathbf{u},\mathbf{x})=\omega(\mathbf{v},\mathbf{w})=\omega(\mathbf{v},\mathbf{x})=\omega(\mathbf{w},\mathbf{x})=0. \tag{26}$$

Expanding $\mathbf{z}_1,\ldots,\mathbf{z}_6$ to low order in t using equations (13)–(15), we find

$$\begin{aligned}
\mathbf{z}_1(t)&=\mathbf{v}+\mathbf{w}+O(t), & \mathbf{z}_2(t)&=\mathbf{v}-\mathbf{w}+O(t),\\
\mathbf{z}_3(t)&=\mathbf{x}+O(t), & \mathbf{z}_4(t)&=\mathbf{u}+t\,\mathbf{u}\times\mathbf{x}+O(t^2),\\
\mathbf{z}_5(t)&=2t\,\mathbf{u}\times\mathbf{w}+O(t^2), & \mathbf{z}_6(t)&=t^2\,\mathbf{u}\times(\mathbf{u}\times\mathbf{w})+O(t^3),
\end{aligned}$$

for small t.

Now calculating using (12) shows that

$$\begin{aligned}
\mathbf{u}\times(\mathbf{u}\times\mathbf{w})&=\tfrac14\big(g(\mathbf{u},\mathbf{w})+i\omega(\mathbf{u},\mathbf{w})\big)\mathbf{u}-\tfrac14\mathbf{w}\\
&=\tfrac14 g(\mathbf{u},\mathbf{w})\mathbf{u}-\tfrac14\mathbf{w},
\end{aligned}\tag{27}$$

as $\omega(\mathbf{u},\mathbf{w})=0$ by (26). Substituting the above expressions for $\mathbf{z}_j(t)$ into (24) and using (27) to rewrite the $\mathbf{u}\times(\mathbf{u}\times\mathbf{w})$ term in $\mathbf{z}_6(t)$, we find that

$$\begin{aligned}
\Phi(y_1,y_2,t)=&\big(y_1+\tfrac14 g(\mathbf{u},\mathbf{w})t^2\big)\,\mathbf{u}+y_1^2\,\mathbf{v}+\big(y_2^2-\tfrac14|\mathbf{u}|^2t^2\big)\,\mathbf{w}\\
&+y_1y_2\,\mathbf{x}+y_1t\,\mathbf{u}\times\mathbf{x}+2y_2t\,\mathbf{u}\times\mathbf{w}\\
&+\text{third-order terms and above in } y_1,y_2,t.
\end{aligned}\tag{28}$$

This is the expansion of Φ up to second order in y_1,y_2,t. Notice that the only first-order term is $y_1\,\mathbf{u}$.

Stratifying terms by their order in y_1, y_2, t is probably not the most helpful view. Instead, it may be better to regard y_1 as having twice the order of y_2 and t. To see this, observe from (28) that

$$\begin{aligned}\Phi(\epsilon^2 y_1, \epsilon y_2, \epsilon t) = \epsilon^2 \Big(&\big(y_1 + \tfrac{1}{4} g(\mathbf{u}, \mathbf{w}) t^2\big)\, \mathbf{u} + \big(y_2^2 - \tfrac{1}{4}|\mathbf{u}|^2 t^2\big)\, \mathbf{w} \\ &+ 2y_2 t\, \mathbf{u} \times \mathbf{w}\Big) + O(\epsilon^3),\end{aligned} \tag{29}$$

for small ϵ. That is, the dominant terms are those in y_1, y_2^2, ty_2 and t^2.

Suppose $\mathbf{u}$ and $\mathbf{w}$ are linearly independent, which is true in the generic case. Then as $\omega(\mathbf{u}, \mathbf{w}) = 0$ we see that $\mathbf{u}, \mathbf{w}$ and $\mathbf{u} \times \mathbf{w}$ are linearly independent and span a special Lagrangian $\mathbb{R}^3$ in $\mathbb{C}^3$. So (29) shows that near 0 to lowest order, N is the image of a map $\mathbb{R}^3 \to \mathbb{R}^3$ given by

$$(y_1, y_2, t) \longmapsto \big(y_1 + \tfrac{1}{4} g(\mathbf{u}, \mathbf{w}) t^2, y_2^2 - \tfrac{1}{4}|\mathbf{u}|^2 t^2, 2y_2 t\big),$$

with image in a special Lagrangian 3-plane. This map is a *double cover* of $\mathbb{R}^3$, branched over the x-axis.

Therefore, if $\mathbf{u}, \mathbf{w}$ are linearly independent then near zero to lowest order, N looks like a branched double cover of a special Lagrangian 3-plane, branched over a real line. To understand how N deviates from this SL 3-plane to leading order, we would need to know the terms in ϵ^3 in (29), which come from the terms in $y_1 y_2, y_1 t, y_2^2 t, y_2 t^2$ and t^3 in the expansion of Φ. These can be calculated by the method above, but we will not write them down as they are rather complex.

Most of the singularities of SL m-folds that we have met so far in [3], [4] and [5] have been *conical* singularities; that is, to lowest order the submanifold resembles a cone with an *isolated* singularity at 0. These singularities follow a different pattern. To lowest order they do resemble a (degenerate) cone, the double SL 3-plane, but this cone should be regarded as *singular* along the branch locus, so that the singularity is not isolated. The interesting information about the singularity of N is not just the cone itself, but also how N deviates from the cone to leading order.

We assumed above that $\mathbf{u}$ and $\mathbf{w}$ are linearly independent. Here is what happens if they are not.

- If $\mathbf{u} = 0$ then $\mathbf{z}_4 \equiv \mathbf{z}_5 \equiv \mathbf{z}_6 \equiv 0$, and Φ reduces to

  $$\Phi(y_1, y_2, t) = \tfrac{1}{2}(y_1^2 + y_2^2)\, \mathbf{z}_1(t) + \tfrac{1}{2}(y_1^2 - y_2^2)\, \mathbf{z}_2(t) + y_1 y_2\, \mathbf{z}_3(t).$$

  The image $N = \operatorname{Image}\Phi$ is a *cone* with singularity at 0, and may be written

  $$N = \big\{x_1\, \mathbf{z}_1(t) + x_2\, \mathbf{z}_2(t) + x_3\, \mathbf{z}_3(t) : x_1, x_2, x_3, t \in \mathbb{R}, \quad x_1^2 = x_2^2 + x_3^2\big\}.$$

  Cones of this kind were studied in [5], in particular in [5, §6].

- If $\mathbf{u} \neq 0$ and $\mathbf{w} = 0$, we find that $\mathbf{z}_1 \equiv \mathbf{z}_2$ for all t, that $\mathbf{z}_3$ is constant, and that $\mathbf{z}_5 \equiv \mathbf{z}_6 \equiv 0$. This situation will be studied in case (a) of §8.3, and an explicit expression for N is given in (38). It turns out that N is a subset of a nonsingular product SL 3-fold $\Sigma \times \mathbb{R}$ in $\mathbb{C}^2 \times \mathbb{C}$, and the 'singularity' is due to a poor choice of coordinates.

- If $\mathbf{u}, \mathbf{w}$ are nonzero but linearly dependent, we need to expand Φ up to third order in y_1, y_2, t and perform a similar analysis to the above. It turns out that if $\mathbf{u}, \mathbf{x}$ are linearly independent, then to lowest order N flattens itself onto the special Lagrangian 3-plane $\langle \mathbf{u}, \mathbf{x}, \mathbf{u} \times \mathbf{x} \rangle_{\mathbb{R}}$ near zero, and is triply branched over the real line $\langle \mathbf{u} \rangle_{\mathbb{R}}$. The details are complicated.

We conclude with two further remarks. Firstly, when $\mathbf{z}_5(0) = 0$ as above we have $\Phi(0, y_2, 0) = \Phi(0, -y_2, 0)$ for all $y_2 \in \mathbb{R}$. This means that away from the singularity $\Phi(0,0,0)$, the SL 3-fold N intersects itself in a half-line, and so is not embedded. This applies to every singular N coming from Theorem 5.1, as any such N can be transformed under the $\mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$-action and translation in t to have $\mathbf{z}_5(0) = 0$.

Secondly, one could extend the ideas above to study a certain class of singularities of SL 3-folds in a very explicit way. Consider SL 3-folds N that are the images of real analytic maps $\Phi : \mathbb{R}^3 \to \mathbb{C}^3$, defined near 0 in $\mathbb{R}^3$, and such that $\mathrm{d}\Phi|_{(0,0,0)}$ is not injective. Generically N is singular at $\Phi(0,0,0)$. By expanding Φ as a power series about 0 and considering only low order terms, one could hopefully construct local models for singularities of SL 3-folds.

7 Division into cases using $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$

In order to study and understand the family of SL 3-folds N constructed in Theorem 5.1, we shall find it helpful to divide the solutions into cases according to the behaviour of the solutions $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 : \mathbb{R} \to \mathbb{C}^3$ of (7) and (13). For $t \in \mathbb{R}$, consider the real vector subspace $\langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t) \rangle_{\mathbb{R}}$ in $\mathbb{C}^3$, where $\langle \dots \rangle_{\mathbb{R}}$ is the span over $\mathbb{R}$.

It turns out that one of the most important things determining the behaviour of solutions $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ of (13) is the dimension of this vector space for generic t. Clearly the dimension lies between 0 and 3, so we divide into four cases:

(i) $\dim \langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t) \rangle_{\mathbb{R}} = 0$ for generic $t \in \mathbb{R}$,

(ii) $\dim \langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t) \rangle_{\mathbb{R}} = 1$ for generic $t \in \mathbb{R}$,

(iii) $\dim \langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t) \rangle_{\mathbb{R}} = 2$ for generic $t \in \mathbb{R}$, and

(iv) $\dim \langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t) \rangle_{\mathbb{R}} = 3$ for generic $t \in \mathbb{R}$.

It can be shown that in cases (i)–(iii) the dimension is constant for all t. But in case (iv), the dimension can drop to 2 for isolated $t \in \mathbb{R}$. The complexity of the solutions to (13)–(15) increases with the dimension. Thus cases (i) and (ii) are very straightforward, and we shall discuss them in the rest of the section.

Case (iii) will be the subject of §8. Case (iv) is the most complicated, and will be divided into subcases and discussed in §9–§11.

Case (i).

In this case we have $\mathbf{z}_1 \equiv \mathbf{z}_2 \equiv \mathbf{z}_3 \equiv 0$ for all t. Thus $\mathbf{z}_4$ and $\mathbf{z}_5$ are constant by (14), and (15) integrates to $\mathbf{z}_6(t) = t\,\mathbf{z}_4 \times \mathbf{z}_5 + \mathbf{z}_6(0)$. So solutions exist for all $t \in \mathbb{R}$, and the SL 3-fold N of (16) is

$$N = \bigl\{ y_1\,\mathbf{z}_4 + y_2\,\mathbf{z}_5 + t\,\mathbf{z}_4 \times \mathbf{z}_5 + \mathbf{z}_6(0) : y_1, y_2, t \in \mathbb{R} \bigr\}.$$

If $\mathbf{z}_4, \mathbf{z}_5$ are linearly independent and $\omega(\mathbf{z}_4, \mathbf{z}_5) = 0$, as in (10), this is an affine special Lagrangian $\mathbb{R}^3$ in $\mathbb{C}^3$. If $\mathbf{z}_4, \mathbf{z}_5$ are not linearly independent, it is a line or a point.

Case (ii).

In this case $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ are proportional, so $\mathbf{z}_2 \times \mathbf{z}_3 \equiv \mathbf{z}_1 \times \mathbf{z}_3 \equiv \mathbf{z}_1 \times \mathbf{z}_2 \equiv 0$, and thus $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ are constant by (13). Let $\mathbf{z}_j = c_j \mathbf{v}$ for $j = 1, 2, 3$, where $c_j \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{C}^3$ is a unit vector. Define a quadratic polynomial $Q(y_1, y_2)$ by

$$Q(y_1, y_2) = \tfrac{1}{2} c_1 (y_1^2 + y_2^2) + \tfrac{1}{2} c_2 (y_1^2 - y_2^2) + c_3 y_1 y_2.$$

Then from (16) we have

$$N = \bigl\{ y_1\,\mathbf{z}_4(t) + y_2\,\mathbf{z}_5(t) + Q(y_1, y_2)\mathbf{v} + \mathbf{z}_6(t) : y_1, y_2, t \in \mathbb{R} \bigr\}.$$

Define a quadric P' in $\mathbb{R}^3$ to be $\bigl\{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_3 = Q(y_1, y_2)\bigr\}$, and for each $t \in \mathbb{R}$ define an affine map $\phi'_t : \mathbb{R}^3 \to \mathbb{C}^3$ by

$$\phi'_t : (y_1, y_2, y_3) \mapsto y_1\,\mathbf{z}_4(t) + y_2\,\mathbf{z}_5(t) + y_3\,\mathbf{v} + \mathbf{z}_6(t).$$

Then N may be written $\bigl\{\phi'_t(p') : t \in \mathbb{R},\ p' \in P'\bigr\}$. That is, N is the total space of a family of quadrics $\phi'_t(P')$, which live in affine Lagrangian 3-planes $\mathbb{R}^3$ in $\mathbb{C}^3$.

Now in [5] we constructed SL m-folds with this property, and it is easy to show that the SL 3-folds of case (ii) above were studied in [5, §7], in particular in [5, Ex. 7.4 & Ex. 7.5]. By the classification of quadratic forms, as Q is nonzero it is equivalent under a linear transformation of $\mathbb{R}^2$ to one of the standard forms

$$y_1^2 + y_2^2, \quad -y_1^2 - y_2^2, \quad y_1^2 - y_2^2 \quad \text{and} \quad y_1^2.$$

In the first two cases the 3-fold N of (16) is one of those constructed in [5, Ex. 7.4], in the third it is one of those constructed in [5, Ex. 7.5], and in the fourth N is reducible and splits as a product $N' \times \mathbb{R}$ in $\mathbb{C}^2 \times \mathbb{C}$, where N' is an SL 2-fold in $\mathbb{C}^2$.

8 Case (iii) of §7

We shall now study the SL 3-folds in $\mathbb{C}^3$ constructed in Theorem 5.1 corresponding to case (iii) of §7. In §8.1 we show that any $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ satisfying (7),

(13) and case (iii) of §7 are equivalent under the natural symmetry group of the construction to $\mathbf{z}_1' = \mathbf{z}_2' = (\mathrm{e}^{it}, -i\mathrm{e}^{-it}, 0)$ and $\mathbf{z}_3' = (0,0,1)$.

Then in §8.2 we solve the remaining equations (8)–(10) and (14)–(15) for $\mathbf{z}_4, \mathbf{z}_5$ and $\mathbf{z}_6$, with these values of $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$. This leads to the main result of this section, Theorem 8.4, which gives an explicit family of SL 3-folds in $\mathbb{C}^3$. Section 8.3 then discusses these 3-folds, doing a parameter count, showing that they are all ruled by straight lines, and studying the periodic solutions.

8.1 Solving the equations for $\mathbf{z}_1, \mathbf{z}_2$ and $\mathbf{z}_3$

As in case (iii) of §7, suppose that $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 : \mathbb{R} \to \mathbb{C}^3$ are solutions of (7) and (13) such that $\langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t) \rangle_{\mathbb{R}}$ has dimension 2 for generic $t \in \mathbb{R}$. This dimension must be less than or equal to 2 for all $t \in \mathbb{R}$, by upper semicontinuity of dimension. But if the dimension is 0 or 1 for any t then it is so for all, since $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ are constant as in parts (i) and (ii) of §7. Thus $\dim \langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t) \rangle_{\mathbb{R}} = 2$ for *all* $t \in \mathbb{R}$.

Now in §5.1 we defined an action of $\mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ on the set of solutions $\mathbf{z}_1, \ldots, \mathbf{z}_6$ to (13)–(15). If we restrict our attention to $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ then e and f play no rôle, and the group acting is $\mathrm{GL}(2,\mathbb{R})$. We shall use this $\mathrm{GL}(2,\mathbb{R})$-action to write $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ in a convenient form.

Proposition 8.1 *Let $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ be as above. Then they are equivalent under the* $\mathrm{GL}(2,\mathbb{R})$*-action described in Proposition 5.2 to $\mathbf{z}_1', \mathbf{z}_2', \mathbf{z}_3' : \mathbb{R} \to \mathbb{C}^3$ with $\mathbf{z}_1'(0) = \mathbf{z}_2'(0)$ and $\left|\mathbf{z}_3'(0)\right| = 1$.*

Proof. As $\mathbf{z}_1(0), \mathbf{z}_2(0), \mathbf{z}_3(0)$ span a vector space of dimension 2, they satisfy $a_1 \mathbf{z}_1(0) + a_2 \mathbf{z}_2(0) + a_3 \mathbf{z}_3(0) = 0$ for some $a_1, a_2, a_3 \in \mathbb{R}$ not all zero. We may think of the $\mathbf{z}_j(0)$ as giving a linear map $S^2\mathbb{R}^2 \to \mathbb{C}^3$ with kernel $\mathbb{R}$, and so (a_1, a_2, a_3) defines a point in the kernel in $S^2\mathbb{R}^2$. Under the action of $\mathrm{GL}(2,\mathbb{R})$ on $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ defined in (18)–(20), the vector (a_1, a_2, a_3) transforms under the natural action of $\mathrm{GL}(2,\mathbb{R})$ on $\mathbb{R}^3 = S^2\mathbb{R}^2$.

It can be shown that $\mathrm{GL}(2,\mathbb{R})$ acts on $\mathbb{R}^3$ as the group preserving the *Lorentzian conformal structure* with null cone $a_1^2 - a_2^2 - a_3^2 = 0$. Thus there are three $\mathrm{GL}(2,\mathbb{R})$-orbits of nonzero vectors in $\mathbb{R}^3$: the 'time-like' vectors with $a_1^2 - a_2^2 - a_3^2 > 0$, the 'space-like' vectors with $a_1^2 - a_2^2 - a_3^2 < 0$, and the 'null' vectors with $a_1^2 - a_2^2 - a_3^2 = 0$.

Therefore every nonzero vector (a_1, a_2, a_3) is equivalent under the $\mathrm{GL}(2,\mathbb{R})$-action to $(1,0,0)$ or $(0,1,0)$ or $(1,-1,0)$. Hence we can transform $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ under the $\mathrm{GL}(2,\mathbb{R})$ action to $\mathbf{z}_1', \mathbf{z}_2', \mathbf{z}_3'$ satisfying either (a) $\mathbf{z}_1'(0) = 0$, (b) $\mathbf{z}_2'(0) = 0$, or (c) $\mathbf{z}_1'(0) - \mathbf{z}_2'(0) = 0$. We shall show that (a) and (b) contradict case (iii) of §7, which leaves case (c).

In part (a), as $\mathbf{z}_2'(0), \mathbf{z}_3'(0)$ are linearly independent and $\omega\bigl(\mathbf{z}_2'(0), \mathbf{z}_3'(0)\bigr) = 0$ by (7), we see that $\mathbf{z}_2'(0), \mathbf{z}_3'(0)$ and $\mathbf{z}_2'(0) \times \mathbf{z}_3'(0)$ are linearly independent. But (13) gives

$$\mathbf{z}_1'(t) = 2t\, \mathbf{z}_2'(0) \times \mathbf{z}_3'(0) + O(t^3), \quad \mathbf{z}_2'(t) = \mathbf{z}_2'(0) + O(t^2), \quad \mathbf{z}_3'(t) = \mathbf{z}_3'(0) + O(t^2)$$

for small t. Therefore $\mathbf{z}'_1(t), \mathbf{z}'_2(t)$ and $\mathbf{z}'_3(t)$ are linearly independent for small, nonzero t. This contradicts (iii). Similarly, part (b) leads to a contradiction.

Thus part (c) holds, and we can transform the $\mathbf{z}_j$ to $\mathbf{z}'_j$ with $\mathbf{z}'_1(0) = \mathbf{z}'_2(0)$. Clearly $\mathbf{z}'_3(0)$ must be nonzero for case (iii) to hold. We can then use a dilation in $\mathrm{GL}(2,\mathbb{R})$ to rescale the $\mathbf{z}'_j$ to get $\left|\mathbf{z}'_3(0)\right| = 1$. This completes the proof. $\square$

Apart from the $\mathrm{GL}(2,\mathbb{R})$-action considered above, the construction is also invariant under the action of $\mathrm{SU}(3)$ on $\mathbb{C}^3$. Given that $\left|\mathbf{z}'_3(0)\right| = 1$, we can apply an $\mathrm{SU}(3)$ transformation to fix $\mathbf{z}'_3(0) = (0,0,1)$. Thus we see that any solution $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ of case (iii) can be transformed under the natural symmetry groups $\mathrm{GL}(2,\mathbb{R})$, $\mathrm{SU}(3)$ of the construction to a new solution $\mathbf{z}'_1, \mathbf{z}'_2, \mathbf{z}'_3$ with $\mathbf{z}'_1(0) = \mathbf{z}'_2(0)$ and $\mathbf{z}'_3(0) = (0,0,1)$.

We can now solve for $\mathbf{z}'_1, \mathbf{z}'_2$ and $\mathbf{z}'_3$ explicitly. By symmetry between $\mathbf{z}_1$ and $\mathbf{z}_2$ in the first two equations of (13) we see that $\mathbf{z}'_1(t) = \mathbf{z}'_2(t)$ for all t. Also, $\mathrm{d}\mathbf{z}'_3/\mathrm{d}t = 0$ by the third equation of (13), so that $\mathbf{z}'_3(t) = (0,0,1)$ for all t.

Set $\mathbf{z}'_1 = \mathbf{z}'_2 = (w_1, w_2, w_3)$, for differentiable functions $w_1, w_2, w_3 : \mathbb{R} \to \mathbb{C}$. Then equations (13) become

$$\frac{\mathrm{d}}{\mathrm{d}t}(w_1, w_2, w_3) = 2(w_1, w_2, w_3) \times (0,0,1) = (\bar{w}_2, -\bar{w}_1, 0),$$

so that

$$\frac{\mathrm{d}w_1}{\mathrm{d}t} = \bar{w}_2, \quad \frac{\mathrm{d}w_2}{\mathrm{d}t} = -\bar{w}_1 \quad \text{and} \quad \frac{\mathrm{d}w_3}{\mathrm{d}t} = 0.$$

The first two equations give $\frac{\mathrm{d}^2 w_1}{\mathrm{d}t^2} = -w_1$, which has solutions $w_1(t) = X'\mathrm{e}^{it} + Y'\mathrm{e}^{-it}$ for $X', Y' \in \mathbb{C}$, and so $w_2(t) = i\bar{Y}'\mathrm{e}^{it} - i\bar{X}'\mathrm{e}^{-it}$. The third equation has solutions $w_3(t) = Z'$ for $Z' \in \mathbb{C}$. But we need the solutions to satisfy (7), which reduces to $\operatorname{Im} Z' = 0$.

Thus $\mathbf{z}'_1, \mathbf{z}'_2$ and $\mathbf{z}'_3$ are given by

$$\mathbf{z}'_1(t) = \mathbf{z}'_2(t) = (X'\mathrm{e}^{it} + Y'\mathrm{e}^{-it}, i\bar{Y}'\mathrm{e}^{it} - i\bar{X}'\mathrm{e}^{-it}, Z') \quad \text{and} \quad \mathbf{z}'_3(t) = (0,0,1),$$

with $X', Y' \in \mathbb{C}$ and $Z' \in \mathbb{R}$. The condition that $\dim\langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)\rangle_{\mathbb{R}} = 2$ is satisfied, for all t, if X' and Y' are not both zero.

Although we have used both the $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{SU}(3)$ actions to bring the $\mathbf{z}'_j$ to this special form, we have not used all the freedom in these two group actions, and we can use what remains to choose the values of X', Y' and Z'. From (18)–(20), the subgroup of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}(2,\mathbb{R})$ preserving the condition $\mathbf{z}_1 = \mathbf{z}_2$ and fixing $\mathbf{z}_3$ are those with $b = 0$ and $ad = 1$, so that $\delta = 1$.

Thus, putting $b = 0$ and $d = a^{-1}$, by (18)–(20) we transform $\mathbf{z}'_1, \mathbf{z}'_2, \mathbf{z}'_3$ to

$$\mathbf{z}''_1(t) = \mathbf{z}''_2(t) = a^2\mathbf{z}'_1(t) + ac(0,0,1) \quad \text{and} \quad \mathbf{z}''_3(t) = (0,0,1).$$

Choose $a = (|X'|^2 + |Y'|^2)^{-1/2}$ and $c = -a^{-1}Z'$. Then we get

$$\mathbf{z}''_1(t) = \mathbf{z}''_2(t) = (X''\mathrm{e}^{it} + Y''\mathrm{e}^{-it}, i\bar{Y}''\mathrm{e}^{it} - i\bar{X}''\mathrm{e}^{-it}, 0) \quad \text{and} \quad \mathbf{z}''_3(t) = (0,0,1),$$

where $|X''|^2 + |Y''|^2 = 1$. Finally, applying the SU(3) matrix

$$\begin{pmatrix} \bar{X}'' & -iY'' & 0 \\ -i\bar{Y}'' & X'' & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we transform the $\mathbf{z}_j''$ to $\mathbf{z}_j'''$, where

$$\mathbf{z}_1'''(t) = \mathbf{z}_2'''(t) = (\mathrm{e}^{it}, -i\mathrm{e}^{-it}, 0) \quad \text{and} \quad \mathbf{z}_3'''(t) = (0,0,1).$$

Hence we have proved:

Theorem 8.2 *Any $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 : \mathbb{R} \to \mathbb{C}^3$ satisfying* (7), (13) *and part* (iii) *of* §7 *are equivalent under the natural actions of* $\mathrm{GL}(2,\mathbb{R})$ *and* SU(3) *to*

$$\hat{\mathbf{z}}_1(t) = \hat{\mathbf{z}}_2(t) = (\mathrm{e}^{it}, -i\mathrm{e}^{-it}, 0) \quad \textit{and} \quad \hat{\mathbf{z}}_3(t) = (0,0,1). \tag{30}$$

8.2 Solving the equations for $\mathbf{z}_4, \mathbf{z}_5$ and $\mathbf{z}_6$

Now let us fix $\mathbf{z}_1(t) = \mathbf{z}_2(t) = (\mathrm{e}^{it}, -i\mathrm{e}^{-it}, 0)$ and $\mathbf{z}_3(t) = (0,0,1)$, as in Theorem 8.2, and solve (14) and (15) for $\mathbf{z}_4, \mathbf{z}_5$ and $\mathbf{z}_6$. Write $\mathbf{z}_4 = (p_1, p_2, p_3)$ and $\mathbf{z}_5 = (q_1, q_2, q_3)$ for differentiable functions $p_j, q_j : \mathbb{R} \to \mathbb{C}$. Then the first and second equations of (14) become

$$\begin{gathered} \frac{\mathrm{d}p_1}{\mathrm{d}t} = i\mathrm{e}^{it}\bar{q}_3 + \tfrac{1}{2}\bar{p}_2, \qquad \frac{\mathrm{d}p_2}{\mathrm{d}t} = -\mathrm{e}^{-it}\bar{q}_3 - \tfrac{1}{2}\bar{p}_1 \\ \text{and} \qquad \frac{\mathrm{d}p_3}{\mathrm{d}t} = -i\mathrm{e}^{it}\bar{q}_1 + \mathrm{e}^{-it}\bar{q}_2, \end{gathered} \tag{31}$$

$$\frac{\mathrm{d}q_1}{\mathrm{d}t} = -\tfrac{1}{2}\bar{q}_2, \quad \frac{\mathrm{d}q_2}{\mathrm{d}t} = \tfrac{1}{2}\bar{q}_1 \quad \text{and} \quad \frac{\mathrm{d}q_3}{\mathrm{d}t} = 0. \tag{32}$$

The solutions of (32) are easily shown to be

$$q_1(t) = A\mathrm{e}^{it/2} + B\mathrm{e}^{-it/2}, \;\; q_2(t) = i\bar{A}\mathrm{e}^{-it/2} - i\bar{B}\mathrm{e}^{it/2}, \;\; q_3(t) = C, \tag{33}$$

for $A, B, C \in \mathbb{C}$. Substituting these into (31) gives

$$\begin{aligned} \frac{\mathrm{d}p_1}{\mathrm{d}t} &= i\bar{C}\mathrm{e}^{it} + \tfrac{1}{2}\bar{p}_2, \qquad \frac{\mathrm{d}p_2}{\mathrm{d}t} = -\bar{C}\mathrm{e}^{-it} - \tfrac{1}{2}\bar{p}_1 \\ \text{and} \quad \frac{\mathrm{d}p_3}{\mathrm{d}t} &= -iA\mathrm{e}^{-it/2} - i\bar{A}\mathrm{e}^{it/2} + iB\mathrm{e}^{-3it/2} - i\bar{B}\mathrm{e}^{3it/2}. \end{aligned}$$

The first two reduce to $\frac{\mathrm{d}^2 p_1}{\mathrm{d}t^2} + \frac{1}{4}p_1 = -(\frac{1}{2}C + \bar{C})\mathrm{e}^{it}$, and are then easily solved, and the third gives p_3 by integration. Thus the solutions are

$$\begin{aligned} p_1(t) &= \big(\tfrac{2}{3}C + \tfrac{4}{3}\bar{C}\big)\mathrm{e}^{it} + D\mathrm{e}^{it/2} + E\mathrm{e}^{-it/2}, \\ p_2(t) &= -i\big(\tfrac{2}{3}C + \tfrac{4}{3}\bar{C}\big)\mathrm{e}^{-it} - i\bar{D}\mathrm{e}^{-it/2} + i\bar{E}\mathrm{e}^{it/2}, \\ p_3(t) &= 2A\mathrm{e}^{-it/2} - 2\bar{A}\mathrm{e}^{it/2} - \tfrac{2}{3}B\mathrm{e}^{-3it/2} - \tfrac{2}{3}\bar{B}\mathrm{e}^{3it/2} + F, \end{aligned}$$

for $D,E,F\in\mathbb{C}$. This gives us the full solutions $\mathbf{z}_4,\mathbf{z}_5$ to (14).

Now $\mathbf{z}_4$ and $\mathbf{z}_5$ are required to satisfy equations (8)–(10). Calculation with the formulae above shows that

$$\begin{aligned}\omega(\mathbf{z}_1,\mathbf{z}_5)+\omega(\mathbf{z}_2,\mathbf{z}_5)-\omega(\mathbf{z}_3,\mathbf{z}_4)&=-\operatorname{Im}F,\\ -\omega(\mathbf{z}_1,\mathbf{z}_4)+\omega(\mathbf{z}_2,\mathbf{z}_4)+\omega(\mathbf{z}_3,\mathbf{z}_5)&=\operatorname{Im}C,\\ \text{and}\quad \omega(\mathbf{z}_4,\mathbf{z}_5)&=\operatorname{Im}(2A\bar D+2B\bar E+C\bar F).\end{aligned}$$

Thus (8)–(10) hold if and only if $\operatorname{Im}C=\operatorname{Im}F=\operatorname{Im}(2A\bar D+2B\bar E+C\bar F)=0$.

The two automatic solutions to (14) given by Corollary 5.3 correspond to $\operatorname{Re}C$ and $\operatorname{Re}F$, with $e=\operatorname{Re}C$ and $f=\operatorname{Re}F$. Therefore, as in §5.1, we can use translational symmetry in $\mathbb{R}^2$ to set $\operatorname{Re}C=\operatorname{Re}F=0$ without reducing the set of SL 3-folds we construct. Thus, without loss of generality we can set $C=F=0$, and then (8)–(10) hold if and only if $\operatorname{Im}(A\bar D+B\bar E)=0$.

We summarize our progress so far in the following proposition.

Proposition 8.3 *Define* $\mathbf{z}_1,\dots,\mathbf{z}_5:\mathbb{R}\to\mathbb{C}^3$ *by*

$$\mathbf{z}_1(t)=\mathbf{z}_2(t)=(\mathrm{e}^{it},-i\mathrm{e}^{-it},0),\quad \mathbf{z}_3(t)=(0,0,1),\tag{34}$$

$$\begin{aligned}\mathbf{z}_4(t)=\bigl(&D\mathrm{e}^{it/2}+E\mathrm{e}^{-it/2},-i\bar D\mathrm{e}^{-it/2}+i\bar E\mathrm{e}^{it/2},\\ &2A\mathrm{e}^{-it/2}-2\bar A\mathrm{e}^{it/2}-\tfrac{2}{3}B\mathrm{e}^{-3it/2}-\tfrac{2}{3}\bar B\mathrm{e}^{3it/2}\bigr),\end{aligned}\tag{35}$$

$$\textit{and}\quad \mathbf{z}_5(t)=\bigl(A\mathrm{e}^{it/2}+B\mathrm{e}^{-it/2},i\bar A\mathrm{e}^{-it/2}-i\bar B\mathrm{e}^{it/2},0\bigr),\tag{36}$$

where A,B,D,E *are complex numbers with* $\operatorname{Im}(A\bar D+B\bar E)=0$*. Then* $\mathbf{z}_1,\dots,\mathbf{z}_5$ *satisfy* (7)–(10) *and* (13)–(14).

Next we solve (14) for $\mathbf{z}_6$. Substituting (35) and (36) into (14) gives a rather complicated formula for $\mathrm{d}\mathbf{z}_6/\mathrm{d}t$, which may be integrated in the normal way. We find that $\mathbf{z}_6=(r_1,r_2,r_3)$, where

$$\begin{aligned}r_1(t)&=-\tfrac{1}{6}A\bar B\mathrm{e}^{2it}+(A\bar A+\tfrac{1}{3}B\bar B)\mathrm{e}^{it}-i(A^2+\bar AB)t-\tfrac{2}{3}AB\mathrm{e}^{-it}-\tfrac{1}{6}B^2\mathrm{e}^{-2it}+G,\\ r_2(t)&=\tfrac{i}{6}\bar B^2\mathrm{e}^{2it}-\tfrac{2i}{3}\bar A\bar B\mathrm{e}^{it}+(\bar A^2-A\bar B)t-i(A\bar A+\tfrac{1}{3}B\bar B)\mathrm{e}^{-it}-\tfrac{i}{6}\bar AB\mathrm{e}^{-2it}+H,\\ r_3(t)&=-\tfrac{1}{2}(A\bar E+\bar BD)\mathrm{e}^{it}-i\operatorname{Re}(A\bar D-B\bar E)t-\tfrac{1}{2}(\bar AE+B\bar D)\mathrm{e}^{-it}+I,\end{aligned}$$

for $G,H,I\in\mathbb{C}$. Collecting all the above material together, and setting $G=H=I=0$ for simplicity, we have proved:

Theorem 8.4 *Let* $A,B,D,E\in\mathbb{C}$ *with* $\operatorname{Im}(A\bar D+B\bar E)=0$*, and define* N *to be*

$$\begin{aligned}\Bigl\{\Bigl(&y_1^2\mathrm{e}^{it}+y_1\bigl(D\mathrm{e}^{it/2}+E\mathrm{e}^{-it/2}\bigr)+y_2\bigl(A\mathrm{e}^{it/2}+B\mathrm{e}^{-it/2}\bigr)-\tfrac{1}{6}A\bar B\mathrm{e}^{2it}\\ &+(A\bar A+\tfrac{1}{3}B\bar B)\mathrm{e}^{it}-i(A^2+\bar AB)t-\tfrac{2}{3}AB\mathrm{e}^{-it}-\tfrac{1}{6}B^2\mathrm{e}^{-2it},\\ &-iy_1^2\mathrm{e}^{-it}+iy_1\bigl(-\bar D\mathrm{e}^{-it/2}+\bar E\mathrm{e}^{it/2}\bigr)+iy_2\bigl(\bar A\mathrm{e}^{-it/2}-\bar B\mathrm{e}^{it/2}\bigr)+\tfrac{i}{6}\bar B^2\mathrm{e}^{2it}\\ &-\tfrac{2i}{3}\bar A\bar B\mathrm{e}^{it}+(\bar A^2-A\bar B)t-i(A\bar A+\tfrac{1}{3}B\bar B)\mathrm{e}^{-it}-\tfrac{i}{6}\bar AB\mathrm{e}^{-2it},\\ &y_1y_2+y_1\bigl(2A\mathrm{e}^{-it/2}-2\bar A\mathrm{e}^{it/2}-\tfrac{2}{3}B\mathrm{e}^{-3it/2}-\tfrac{2}{3}\bar B\mathrm{e}^{3it/2}\bigr)-\tfrac{1}{2}(A\bar E+\bar BD)\mathrm{e}^{it}\\ &-i\operatorname{Re}(A\bar D-B\bar E)t-\tfrac{1}{2}(\bar AE+B\bar D)\mathrm{e}^{-it}\Bigr):y_1,y_2,t\in\mathbb{R}\Bigr\}.\end{aligned}\tag{37}$$

Then N is a special Lagrangian 3-fold in $\mathbb{C}^3$. Furthermore, any special Lagrangian 3-fold in $\mathbb{C}^3$ constructed using Theorem 5.1 and satisfying case (iii) *of* §*7 is isomorphic to one of this family under an automorphism of $\mathbb{C}^3$.*

8.3 Discussion

As in §5.2, here is a parameter count for the set of SL 3-folds constructed in Theorem 8.4. The 3-folds depend on 4 complex numbers A, B, D, E, which is 8 real parameters, and they satisfy one real equation $\operatorname{Im}(A\bar{D}+B\bar{E})=0$, reducing the number of parameters to 7. Should we reduce this further to allow for isomorphisms between members of the family?

Well, in §8.1 we used up the $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{SU}(3)$ symmetries to fix $\mathbf{z}_1, \mathbf{z}_2$ and $\mathbf{z}_3$, and in §8.2 we used the $\mathbb{R}^2$ translational symmetry to set $\operatorname{Re} C = \operatorname{Re} F = 0$, and the $\mathbb{C}^3$ translational symmetry to fix $G = H = I = 0$. So the $\mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ and $\mathrm{SU}(3) \ltimes \mathbb{C}^3$ symmetry groups are both already fully accounted for. The only remaining symmetry in the problem is translation in time, $t \mapsto t + c$.

Let $c \in \mathbb{R}$, and replace t, A, B, D, E by t', A', B', D', E', where

$$t' = t + c, \;\; A' = \mathrm{e}^{ic/2}A, \;\; B' = \mathrm{e}^{3ic/2}B, \;\; D' = \mathrm{e}^{ic/2}D, \;\; E' = \mathrm{e}^{3ic/2}E.$$

Then a point (z_1, z_2, z_3) in N is replaced by (z_1', z_2', z_3'), where

$$z_1' = \mathrm{e}^{ic}\bigl(z_1 - i(A^2 + \bar{A}B)c\bigr), \qquad z_2' = \mathrm{e}^{-ic}\bigl(z_2 + (\bar{A}^2 - A\bar{B})c\bigr)$$
$$\text{and} \qquad z_3' = z_3 - i\operatorname{Re}(A\bar{D} - B\bar{E})c.$$

This map $(z_1, z_2, z_3) \mapsto (z_1', z_2', z_3')$ lies in $\mathrm{SU}(3) \ltimes \mathbb{C}^3$.

Therefore if N' is defined as in (37), but using constants $A' = \mathrm{e}^{ic/2}A$, $B' = \mathrm{e}^{3ic/2}B$, $D' = \mathrm{e}^{ic/2}D$ and $E' = \mathrm{e}^{3ic/2}E$, then N' is isomorphic to N under an $\mathrm{SU}(3) \ltimes \mathbb{C}^3$ transformation. So we should reduce the number of real parameters by one, and the family of distinct SL 3-folds from Theorem 8.4 up to automorphisms of $\mathbb{C}^3$ has 6 dimensions. For comparison, the family from Theorem 5.1, of which this is a special case, has 9 dimensions.

Next we consider the conditions for the evolution to be periodic in t. From (37) it is clear that the evolution will be periodic, with period 4π, if and only if the coefficients of t vanish. That is, the evolution is periodic if $A^2 + \bar{A}B = 0$, $\bar{A}^2 - A\bar{B} = 0$ and $\operatorname{Re}(B\bar{E} - A\bar{D}) = 0$. The first two equations hold if and only if $A = 0$, and then the second equation becomes $\operatorname{Re}(B\bar{E}) = 0$. But when $A = 0$ the conditions on A, B, D and E in Theorem 8.4 become $\operatorname{Im}(B\bar{E}) = 0$. Thus $B\bar{E} = 0$.

Hence the evolution is periodic when $A = B\bar{E} = 0$, that is, if either

(a) $A = B = 0$, or

(b) $A = E = 0$.

In case (a), the 3-fold N of (37) is given by

$$\begin{aligned}\Bigl\{&\bigl(y_1^2\mathrm{e}^{it} + y_1(D\mathrm{e}^{it/2} + E\mathrm{e}^{-it/2}),\\ &-iy_1^2\mathrm{e}^{-it} + iy_1(-\bar{D}\mathrm{e}^{-it/2} + \bar{E}\mathrm{e}^{it/2}), y_1y_2\bigr) : y_1, y_2, t \in \mathbb{R}\Bigr\}.\end{aligned} \tag{38}$$

Notice that for each (z_1, z_2, z_3) in N, the third coordinate $z_3 = y_1 y_2$ is *real*. This implies that N is a subset of $\Sigma \times \mathbb{R}$ in $\mathbb{C}^2 \times \mathbb{C}$, where Σ is an SL 2-fold in $\mathbb{C}^2$, which is in fact a *complex parabola* with respect to an alternative complex structure on $\mathbb{C}^2$. However, the parametrization (y_1, y_2, t) does not respect the product structure.

In case (b), the 3-fold N of (37) is given by

$$\begin{aligned}\Big\{\big(&y_1^2 \mathrm{e}^{it} + y_1 D \mathrm{e}^{it/2} + y_2 B \mathrm{e}^{-it/2} + \tfrac{1}{3} B\bar{B} \mathrm{e}^{it} - \tfrac{1}{6} B^2 \mathrm{e}^{-2it},\\ &-iy_1^2 \mathrm{e}^{-it} - iy_1 \bar{D} \mathrm{e}^{-it/2} - iy_2 \bar{B} \mathrm{e}^{it/2} - \tfrac{i}{3} B\bar{B} \mathrm{e}^{-it} + \tfrac{i}{6} \bar{B}^2 \mathrm{e}^{2it},\\ &y_1 y_2 - y_1\big(\tfrac{2}{3} B \mathrm{e}^{-3it/2} + \tfrac{2}{3} \bar{B} \mathrm{e}^{3it/2}\big) - \tfrac{1}{2} \bar{B} D \mathrm{e}^{it} - \tfrac{1}{2} B \bar{D} \mathrm{e}^{-it}\big) : y_1, y_2, t \in \mathbb{R}\Big\}.\end{aligned}$$

Here for each (z_1, z_2, z_3) in N we have $z_2 = -i\bar{z}_1$ and z_3 is real. That is, N is a subset of the special Lagrangian 3-plane

$$\big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_2 = -i\bar{z}_1, \quad z_3 \in \mathbb{R}\big\}.$$

So case (b) is just an unusual parametrization of an $\mathbb{R}^3$ in $\mathbb{C}^3$. Thus, the periodic solutions in Theorem 8.4 are not very interesting.

Finally we note that the SL 3-folds of Theorem 8.4 are ruled by straight lines. Define $\Phi : \mathbb{R}^3 \to \mathbb{C}^3$ as in (24), so that $\Phi(y_1, y_2, t)$ is the vector in (37). Then, as (37) contains no terms in y_2^2, for each fixed $y_1, t \in \mathbb{R}$ the set $\big\{\Phi(y_1, y_2, t) : y_2 \in \mathbb{R}\big\}$ is a real straight line in $\mathbb{C}^3$. So N is fibred by straight lines. Such submanifolds are called *ruled*. Ruled special Lagrangian 3-folds are the subject of [7].

9 Case (iv) of §7

We now move on to study case (iv) of §7, in which $\mathbf{z}_1, \mathbf{z}_2$ and $\mathbf{z}_3$ are linearly independent for generic t. In §9.1 we use the symmetry of the construction to reduce $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ to a convenient form in coordinates, and then in Theorem 9.3 we solve equation (13) explicitly using material from [5, §6].

Section 9.2 rewrites equations (14) and (15) for $\mathbf{z}_4, \mathbf{z}_5$ and $\mathbf{z}_6$ and derives some properties of their solutions, and §9.3 discusses the difficulties of solving them. In two cases of Theorem 9.3 we can solve (14) and (15) explicitly, and we will do this in sections 10 and 11.

9.1 Solving the equations for $\mathbf{z}_1, \mathbf{z}_2$ and $\mathbf{z}_3$

We begin by choosing a convenient form for $\mathbf{z}_1, \mathbf{z}_2$ and $\mathbf{z}_3$ in coordinates, using the symmetry groups $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{SU}(3)$ of the construction, as we did in Theorem 8.2.

Proposition 9.1 *Any* $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 : \mathbb{R} \to \mathbb{C}^3$ *satisfying* (7), (13) *and part* (iv) *of §7 are equivalent under the natural actions of* $\mathrm{SL}(2,\mathbb{R})$ *and* $\mathrm{SU}(3)$ *to*

$$\hat{\mathbf{z}}_1 = (w_1, 0, 0), \quad \hat{\mathbf{z}}_2 = (0, w_2, 0) \quad \textit{and} \quad \hat{\mathbf{z}}_3 = (0, 0, w_3) \tag{39}$$

for differentiable functions $w_1, w_2, w_3 : \mathbb{R} \to \mathbb{C}$.

Proof. Suppose for simplicity that $\mathbf{z}_1(0), \mathbf{z}_2(0)$ and $\mathbf{z}_3(0)$ are linearly independent in $\mathbb{C}^3$. Consider the action of $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{GL}(2,\mathbb{R})$ upon $\mathbf{z}_1, \mathbf{z}_2$ and $\mathbf{z}_3$ given in (18)–(20), where $\delta = ad - bc = 1$. We can think of this as an action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^3$, where (a_1, a_2, a_3) in $\mathbb{R}^3$ is mapped to $a_1\mathbf{z}_1 + a_2\mathbf{z}_2 + a_3\mathbf{z}_3$. This action preserves the quadratic form $a_1^2 - a_2^2 - a_3^2$ upon $\mathbb{R}^3$.

Now there is a second, positive definite quadratic form on $\mathbb{R}^3$ given by

$$(a_1, a_2, a_3) \mapsto \bigl|a_1\mathbf{z}_1(0) + a_2\mathbf{z}_2(0) + a_3\mathbf{z}_3(0)\bigr|^2.$$

By standard results on simultaneous diagonalization of quadratic forms, there is a basis of $\mathbb{R}^3$ in which both quadratic forms are diagonal. Choose an element of $\mathrm{SL}(2,\mathbb{R})$ which takes this basis to vectors proportional to $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Let $\mathbf{z}'_1, \mathbf{z}'_2, \mathbf{z}'_3$ be the transforms of $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ under this element of $\mathrm{SL}(2,\mathbb{R})$. Then the quadratic form

$$(a_1, a_2, a_3) \mapsto \bigl|a_1\mathbf{z}'_1(0) + a_2\mathbf{z}'_2(0) + a_3\mathbf{z}'_3(0)\bigr|^2$$

is diagonal with respect to the standard basis of $\mathbb{R}^3$. That is, $\mathbf{z}'_1(0), \mathbf{z}'_2(0)$ and $\mathbf{z}'_3(0)$ are *orthogonal*. Now from (7) we see that $\mathbf{z}'_1(0), \mathbf{z}'_2(0), \mathbf{z}'_3(0)$ span a Lagrangian plane in $\mathbb{C}^3$. But any three nonzero orthogonal elements in a Lagrangian plane in $\mathbb{C}^3$ are conjugate under a matrix in $\mathrm{SU}(3)$ to $(w_1, 0, 0)$, $(0, w_2, 0)$ and $(0, 0, w_3)$ for some $w_j \in \mathbb{C} \setminus \{0\}$.

Applying this $\mathrm{SU}(3)$ matrix to $\mathbf{z}'_1, \mathbf{z}'_2, \mathbf{z}'_3$ gives $\hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2, \hat{\mathbf{z}}_3$, where (39) holds for $t = 0$. But it is easy to see from (13) that if (39) holds at $t = 0$ then it holds for all t. This completes the proof. $\square$

So suppose that $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ are given by

$$\mathbf{z}_1 = (w_1, 0, 0), \quad \mathbf{z}_2 = (0, w_2, 0) \quad \text{and} \quad \mathbf{z}_3 = (0, 0, w_3) \tag{40}$$

for differentiable functions $w_1, w_2, w_3 : \mathbb{R} \to \mathbb{C}$ which are all nonzero for generic t. Then (7) automatically holds, and the evolution equations (13) are equivalent to the o.d.e.s

$$\frac{\mathrm{d}w_1}{\mathrm{d}t} = \overline{w_2 w_3}, \quad \frac{\mathrm{d}w_2}{\mathrm{d}t} = -\,\overline{w_3 w_1}, \quad \frac{\mathrm{d}w_3}{\mathrm{d}t} = -\,\overline{w_1 w_2}. \tag{41}$$

These are the same as equation (31) of [5, Th. 6.1]. Thus we can use the material of [5, §6] to understand their solutions very explicitly. In particular, [5, Prop. 6.2] and the discussion after it gives

Proposition 9.2 *For any given initial data* $w_1(0), w_2(0), w_3(0)$*, solutions* $w_j(t)$ *of* (41) *exist for all* $t \in \mathbb{R}$*. Wherever the* $w_j(t)$ *are nonzero, these functions may be written*

$$w_1 = \mathrm{e}^{i\theta_1}\sqrt{\alpha_1 + u}, \quad w_2 = \mathrm{e}^{i\theta_2}\sqrt{\alpha_2 - u} \quad \textit{and} \quad w_3 = \mathrm{e}^{i\theta_3}\sqrt{\alpha_3 - u},$$

where $\alpha_j \in \mathbb{R}$ and $u, \theta_1, \theta_2, \theta_3 : \mathbb{R} \to \mathbb{R}$ are differentiable functions. Define

$$Q(u) = (\alpha_1 + u)(\alpha_2 - u)(\alpha_3 - u) \quad \textit{and} \quad \theta = \theta_1 + \theta_2 + \theta_3.$$

Then $Q(u)^{1/2} \sin\theta \equiv A$ for some $A \in \mathbb{R}$, and u and θ_j satisfy

$$\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)^2 = 4\big(Q(u) - A^2\big), \qquad \frac{\mathrm{d}\theta_1}{\mathrm{d}t} = -\frac{A}{\alpha_1 + u},$$
$$\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = \frac{A}{\alpha_2 - u} \qquad \textit{and} \qquad \frac{\mathrm{d}\theta_3}{\mathrm{d}t} = \frac{A}{\alpha_3 - u}.$$

As in [5, §6], the solutions to these equations involve the *Jacobi elliptic functions*, to which we now give a brief introduction. The following material can be found in Chandrasekharan [1, Ch. VII].

For each $k \in [0, 1]$, the Jacobi elliptic functions $\mathrm{sn}(t, k)$, $\mathrm{cn}(t, k)$, $\mathrm{dn}(t, k)$ with modulus k are the unique solutions to the o.d.e.s

$$\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{sn}(t,k)\big)^2 = \big(1 - \mathrm{sn}^2(t,k)\big)\big(1 - k^2\mathrm{sn}^2(t,k)\big),$$
$$\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{cn}(t,k)\big)^2 = \big(1 - \mathrm{cn}^2(t,k)\big)\big(1 - k^2 + k^2\mathrm{cn}^2(t,k)\big),$$
$$\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{dn}(t,k)\big)^2 = -\big(1 - \mathrm{dn}^2(t,k)\big)\big(1 - k^2 - \mathrm{dn}^2(t,k)\big),$$

with initial conditions

$$\mathrm{sn}(0,k) = 0, \quad \mathrm{cn}(0,k) = 1, \quad \mathrm{dn}(0,k) = 1,$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{sn}(0,k) = 1, \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{cn}(0,k) = 0, \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{dn}(0,k) = 0.$$

They satisfy the identities

$$\mathrm{sn}^2(t,k) + \mathrm{cn}^2(t,k) = 1 \quad \text{and} \quad k^2\mathrm{sn}^2(t,k) + \mathrm{dn}^2(t,k) = 1,$$

and the differential equations

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{sn}(t,k) = \mathrm{cn}(t,k)\mathrm{dn}(t,k), \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{cn}(t,k) = -\mathrm{sn}(t,k)\mathrm{dn}(t,k)$$
$$\text{and} \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{dn}(t,k) = -k^2\mathrm{sn}(t,k)\mathrm{cn}(t,k).$$

When $k = 0$ or 1 they reduce to trigonometric functions:

$$\mathrm{sn}(t,0) = \sin t, \quad \mathrm{cn}(t,0) = \cos t, \quad \mathrm{dn}(t,0) = 1, \tag{42}$$
$$\mathrm{sn}(t,1) = \tanh t, \quad \mathrm{cn}(t,1) = \mathrm{dn}(t,1) = \mathrm{sech}\, t. \tag{43}$$

For $k \in [0, 1)$ the Jacobi elliptic functions are periodic in t.

As in [5, §5.4], we choose $\alpha_1, \alpha_2, \alpha_3$ uniquely such that $\alpha_j > 0$ and $\frac{1}{\alpha_1} = \frac{1}{\alpha_2} + \frac{1}{\alpha_3}$. This then implies that Q has a maximum at 0, and $0 \leqslant A^2 \leqslant \alpha_1\alpha_2\alpha_3$. From the results of [5, §5.4] and [5, §6] we can then deduce the following theorem.

Theorem 9.3 *In the situation above, suppose that u has a minimum at $t=0$ and that $\theta_2(0)=\theta_3(0)=0$, $A\geqslant 0$ and $\alpha_2\leqslant\alpha_3$. Then exactly one of the following four cases holds.*

(a) $A=0$ *and* $\alpha_2=\alpha_3$, *and* w_1,w_2,w_3 *are given by*

$$\begin{aligned} &w_1(t)=\sqrt{3\alpha_1}\tanh\bigl(t\sqrt{3\alpha_1}\,\bigr) \qquad \textit{and} \\ &w_2(t)=w_3(t)=\sqrt{3\alpha_1}\,\operatorname{sech}\bigl(t\sqrt{3\alpha_1}\,\bigr). \end{aligned} \tag{44}$$

(b) $A=0$ *and* $\alpha_2<\alpha_3$, *and* w_1,w_2,w_3 *are given by*

$$\begin{gathered} w_1=(\alpha_1+\alpha_2)^{1/2}\mathrm{sn}(\sigma t,\tau), \quad w_2=(\alpha_1+\alpha_2)^{1/2}\mathrm{cn}(\sigma t,\tau) \\ \textit{and} \qquad w_3=(\alpha_1+\alpha_3)^{1/2}\mathrm{dn}(\sigma t,\tau), \end{gathered}$$

where $\sigma=(\alpha_1+\alpha_3)^{1/2}$ *and* $\tau=(\alpha_1+\alpha_2)^{1/2}(\alpha_1+\alpha_3)^{-1/2}$. *Note that* w_1,w_2,w_3 *are periodic in* t.

(c) $0<A<(\alpha_1\alpha_2\alpha_3)^{1/2}$. *Let the roots of* $Q(u)-A^2$ *be* $\gamma_1,\gamma_2,\gamma_3$, *ordered such that* $\gamma_1\leqslant 0\leqslant\gamma_2\leqslant\gamma_3$. *Then* u *and* $\theta_1,\theta_2,\theta_3$ *are given by*

$$\begin{aligned} u(t)&=\gamma_1+(\gamma_2-\gamma_1)\mathrm{sn}^2(\sigma t,\tau), \\ \theta_1(t)&=\theta_1(0)-A\int_0^t\frac{\mathrm{d}s}{\alpha_1+\gamma_1+(\gamma_2-\gamma_1)\mathrm{sn}^2(\sigma s,\tau)}, \\ \theta_2(t)&=A\int_0^t\frac{\mathrm{d}s}{\alpha_2-\gamma_1-(\gamma_2-\gamma_1)\mathrm{sn}^2(\sigma s,\tau)} \\ \textit{and}\quad \theta_3(t)&=A\int_0^t\frac{\mathrm{d}s}{\alpha_3-\gamma_1-(\gamma_2-\gamma_1)\mathrm{sn}^2(\sigma s,\tau)}, \end{aligned}$$

where $\sigma=(\gamma_3-\gamma_1)^{1/2}$ *and* $\tau=(\gamma_2-\gamma_1)^{1/2}(\gamma_3-\gamma_1)^{-1/2}$.

(d) $A=(\alpha_1\alpha_2\alpha_3)^{1/2}$. *Define* $a_1,a_2,a_3\in\mathbb{R}$ *by*

$$a_1=-\alpha_1^{-1/2}(\alpha_2\alpha_3)^{1/2},\ \ a_2=\alpha_2^{-1/2}(\alpha_3\alpha_1)^{1/2},\ \ a_3=\alpha_3^{-1/2}(\alpha_1\alpha_2)^{1/2}.$$

Then $a_1+a_2+a_3=0$, *as* $\frac{1}{\alpha_1}=\frac{1}{\alpha_2}+\frac{1}{\alpha_3}$, *and* w_1,w_2,w_3 *are given by*

$$w_1(t)=i\sqrt{\alpha_1}\,\mathrm{e}^{ia_1t},\quad w_2(t)=\sqrt{\alpha_2}\,\mathrm{e}^{ia_2t}\quad\textit{and}\quad w_3(t)=\sqrt{\alpha_3}\,\mathrm{e}^{ia_3t}.$$

Here we have made a number of assumptions to simplify the formulae, none of which really reduces the generality of the result. Supposing u has a minimum at $t=0$ means that in part (c) we get $u(t)=\gamma_1+(\gamma_2-\gamma_1)\mathrm{sn}^2(\sigma t,\tau)$ rather than $u(t)=\gamma_1+(\gamma_2-\gamma_1)\mathrm{sn}^2(\sigma t+\upsilon,\tau)$ for some $\upsilon\in\mathbb{R}$, and similarly for parts (a) and (b). Thus the assumption can be removed by adding a constant to t.

Setting $\theta_2(0) = \theta_3(0) = 0$ has the effect of tidying up the constant phase factors in w_1, w_2, w_3, so that the w_j are real in cases (a) and (b). We can change the sign of A by replacing w_1 by $-w_1$ and t by $-t$, and we can swap α_2 and α_3 by swapping w_2 and w_3. Thus the assumptions that $A \geqslant 0$ and $\alpha_2 \leqslant \alpha_3$ are no real restriction.

Note that case (a) follows from case (b), as $\tau = 1$ when $\alpha_2 = \alpha_3$, so the sn, cn and dn expressions reduce to tanh and sech by (43). Also, in case (a) the equations $\alpha_2 = \alpha_3$ and $\frac{1}{\alpha_1} = \frac{1}{\alpha_2} + \frac{1}{\alpha_3}$ imply that $\alpha_2 = \alpha_3 = 2\alpha_1$, so we have written the solutions solely in terms of α_1.

9.2 Rewriting the equations for $\mathbf{z}_4, \mathbf{z}_5$ and $\mathbf{z}_6$

Next we shall we rewrite equations (14) and (15). Choose the slightly unusual form

$$\mathbf{z}_4 = (p_1, p_2, q_3) \quad \text{and} \quad \mathbf{z}_5 = (q_1, -q_2, p_3), \tag{45}$$

where $p_j, q_j : (-\epsilon, \epsilon) \to \mathbb{C}$ are differentiable functions. Then (14) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(p_1, p_2, q_3) = (w_1, w_2, 0) \times (q_1, -q_2, p_3) - (0, 0, w_3) \times (p_1, p_2, q_3),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(q_1, -q_2, p_3) = (-w_1, w_2, 0) \times (p_1, p_2, q_3) + (0, 0, w_3) \times (q_1, -q_2, p_3).$$

Expanding using (12), this yields

$$\begin{gathered} \frac{\mathrm{d}p_1}{\mathrm{d}t} = \tfrac{1}{2}(\bar{w}_2\bar{p}_3 + \bar{w}_3\bar{p}_2), \qquad \frac{\mathrm{d}p_2}{\mathrm{d}t} = -\tfrac{1}{2}(\bar{w}_3\bar{p}_1 + \bar{w}_1\bar{p}_3) \\ \text{and} \qquad \frac{\mathrm{d}p_3}{\mathrm{d}t} = -\tfrac{1}{2}(\bar{w}_1\bar{p}_2 + \bar{w}_2\bar{p}_1), \end{gathered} \tag{46}$$

$$\begin{gathered} \frac{\mathrm{d}q_1}{\mathrm{d}t} = \tfrac{1}{2}(\bar{w}_2\bar{q}_3 + \bar{w}_3\bar{q}_2), \qquad \frac{\mathrm{d}q_2}{\mathrm{d}t} = -\tfrac{1}{2}(\bar{w}_3\bar{q}_1 + \bar{w}_1\bar{q}_3) \\ \text{and} \qquad \frac{\mathrm{d}q_3}{\mathrm{d}t} = -\tfrac{1}{2}(\bar{w}_1\bar{q}_2 + \bar{w}_2\bar{q}_1). \end{gathered} \tag{47}$$

These are real linear in the p_j and q_j, and are the same equations, so that any solution to (46) in p_1, p_2, p_3 is also a solution to (47) in q_1, q_2, q_3. Notice also that from (41), $p_j = w_j$ solves (46), and $q_j = w_j$ solves (47). These are the two automatic solutions of (14) we get from Corollary 5.3.

Writing $\mathbf{z}_6 = (r_1, r_2, r_3)$ for $r_j : (-\epsilon, \epsilon) \to \mathbb{C}$, equation (15) becomes

$$\begin{gathered} \frac{\mathrm{d}r_1}{\mathrm{d}t} = \tfrac{1}{2}(\bar{p}_2\bar{p}_3 + \bar{q}_3\bar{q}_2), \qquad \frac{\mathrm{d}r_2}{\mathrm{d}t} = \tfrac{1}{2}(\bar{q}_3\bar{q}_1 - \bar{p}_1\bar{p}_3) \\ \text{and} \qquad \frac{\mathrm{d}r_3}{\mathrm{d}t} = -\tfrac{1}{2}(\bar{p}_1\bar{q}_2 + \bar{p}_2\bar{q}_1), \end{gathered} \tag{48}$$

and integrating these equations gives r_1, r_2 and r_3.

In our next result we work out equations (7)–(10) for $\mathbf{z}_1, \ldots, \mathbf{z}_6$ of this form. The proof is very easy, and we omit it.

Lemma 9.4 *When* $\mathbf{z}_1, \dots, \mathbf{z}_6$ *are defined as above, equation* (7) *holds automatically, and equations* (8)–(10) *are equivalent to*

$$\begin{aligned}\mathrm{Im}(w_1\bar{p}_1 - w_2\bar{p}_2 - w_3\bar{p}_3) &= \mathrm{Im}(w_1\bar{q}_1 - w_2\bar{q}_2 - w_3\bar{q}_3) \\ &= \mathrm{Im}(p_1\bar{q}_1 - p_2\bar{q}_2 - p_3\bar{q}_3) = 0.\end{aligned} \tag{49}$$

Furthermore, if the w_j, p_j *and* q_j *satisfy* (41), (46) *and* (47)*, but not necessarily* (49)*, then* $\mathrm{Im}(w_1\bar{p}_1 - w_2\bar{p}_2 - w_3\bar{p}_3)$, $\mathrm{Im}(w_1\bar{q}_1 - w_2\bar{q}_2 - w_3\bar{q}_3)$ *and* $\mathrm{Im}(p_1\bar{q}_1 - p_2\bar{q}_2 - p_3\bar{q}_3)$ *are constant.*

The last part can be useful in solving equations (46) and (47), once w_1, w_2 and w_3 are chosen, because each solution p_1, p_2, p_3 of (46) gives a conserved quantity in (47), and so reduces the number of real variables by one. Of course, (46) and (47) are really the same, and have the same six-dimensional space of solutions. Thus, given three independent solutions of (46), we may be able to find the other three solutions by a kind of matrix inversion.

The equations simplify further in cases (a) and (b) of Theorem 9.3, when w_1, w_2, w_3 are real. Then the real parts of (46) involve only $\mathrm{Re}(p_j)$, and the imaginary parts of (46) involve only $\mathrm{Im}(p_j)$. So using Lemma 9.4 we deduce:

Lemma 9.5 *In cases* (a) *and* (b) *of Theorem 9.3, the solutions* p_1, p_2, p_3 *of* (46) *are of the form* $p_j = u_j + iv_j$*, where* $u_j, v_j : \mathbb{R} \to \mathbb{R}$ *are differentiable functions satisfying*

$$\begin{gathered}\frac{\mathrm{d}u_1}{\mathrm{d}t} = \tfrac{1}{2}(w_2u_3 + w_3u_2), \qquad \frac{\mathrm{d}u_2}{\mathrm{d}t} = -\tfrac{1}{2}(w_3u_1 + w_1u_3) \\ \textit{and} \qquad \frac{\mathrm{d}u_3}{\mathrm{d}t} = -\tfrac{1}{2}(w_1u_2 + w_2u_1),\end{gathered} \tag{50}$$

$$\begin{gathered}\frac{\mathrm{d}v_1}{\mathrm{d}t} = -\tfrac{1}{2}(w_2v_3 + w_3v_2), \qquad \frac{\mathrm{d}v_2}{\mathrm{d}t} = \tfrac{1}{2}(w_3v_1 + w_1v_3) \\ \textit{and} \qquad \frac{\mathrm{d}v_3}{\mathrm{d}t} = \tfrac{1}{2}(w_1v_2 + w_2v_1).\end{gathered} \tag{51}$$

Furthermore, if (50) *and* (51) *hold, then* $u_1v_1 - u_2v_2 - u_3v_3$ *is constant.*

If we know the full solutions to (50) we can determine the solutions to (51), and vice versa. Here is how. Suppose that u_1^j, u_2^j, u_3^j are solutions to (50) for $j = 1, 2, 3$, and v_1^j, v_2^j, v_3^j solutions to (51) for $j = 1, 2, 3$. Then the last part of the lemma shows that

$$\begin{pmatrix} u_1^1 & u_2^1 & u_3^1 \\ u_1^2 & u_2^2 & u_3^2 \\ u_1^3 & u_2^3 & u_3^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_1^1 & v_1^2 & v_1^3 \\ v_2^1 & v_2^2 & v_2^3 \\ v_3^1 & v_3^2 & v_3^3 \end{pmatrix}$$

is a constant matrix. In particular, if for $j = 1, 2, 3$ the (u_1^j, u_2^j, u_3^j) are linearly independent for some t, then they are linearly independent for all t, and we may

define the v_i^j by

$$\begin{pmatrix} v_1^1 & v_1^2 & v_1^3 \\ v_2^1 & v_2^2 & v_2^3 \\ v_3^1 & v_3^2 & v_3^3 \end{pmatrix} = \begin{pmatrix} u_1^1 & -u_2^1 & -u_3^1 \\ u_1^2 & -u_2^2 & -u_3^2 \\ u_1^3 & -u_2^3 & -u_3^3 \end{pmatrix}^{-1},$$

where the matrix inverse exists. The v_1^j, v_2^j, v_3^j will then be solutions to (51) for $j = 1, 2, 3$, and span the full set of solutions.

9.3 Discussion

So far we have solved equation (13) for $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ explicitly in Theorem 9.3, and we have rewritten equations (14) and (15) for $\mathbf{z}_4, \mathbf{z}_5$ and $\mathbf{z}_6$ in a convenient form, and said something about the properties of their solutions. However, we have not yet solved (14) and (15) explicitly.

In cases (a) and (d) of Theorem 9.3 the author has been able to solve equations (14) and (15) fairly explicitly. These cases will be treated at length in sections 10 and 11 respectively. However, in cases (b) and (c) of Theorem 9.3 the author has made little progress in solving (14) and (15).

In case (c), it may help to split into two cases $\alpha_2 = \alpha_3$ and $\alpha_2 < \alpha_3$. The case $\alpha_2 = \alpha_3$ is simpler, because $w_2 \equiv w_3$, and may be more amenable to explicit solution. In particular, (47) is unchanged by swapping round p_2 and p_3. Thus we can separately consider symmetric solutions with $p_2 = p_3$, and antisymmetric solutions with $p_1 = 0$ and $p_2 = -p_3$.

It would be interesting to know about the *periodicity* of solutions to equations (46)–(48) in cases (b) and (c) of Theorem 9.3. In particular, in case (b) the w_j are periodic with period $T > 0$. If equations (46)–(48) admitted interesting solutions with period nT for some integer $n \geqslant 1$, then the corresponding SL 3-fold N would be a closed, immersed copy of $\mathbb{R}^2 \times \mathcal{S}^1$ rather than $\mathbb{R}^3$.

In fact a weaker form of periodicity would be sufficient, where the map Φ of (24) satisfies $\Phi(y_1 + c_1, y_2 + c_2, t + nT) = \Phi(y_1, y_2, t)$ for $c_1, c_2 \in \mathbb{R}$. We also saw in [5, §6] that many of the solutions w_1, w_2, w_3 in part (c) are periodic, so we can ask the same question.

10 Case (a) of Theorem 9.3

We now study the SL 3-folds N in $\mathbb{C}^3$ which come from the construction of Theorem 5.1 when $\mathbf{z}_1, \mathbf{z}_2$ and $\mathbf{z}_3$ are as in case (a) of Theorem 9.3. For simplicity we set $\alpha_1 = \frac{1}{3}$, so that the functions w_j in case (a) of Theorem 9.3 are $w_1(t) = \tanh t$ and $w_2(t) = w_3(t) = \operatorname{sech} t$. Since we are free to rescale $\alpha_1, \alpha_2, \alpha_3$ using dilations in $\mathrm{GL}(2, \mathbb{R})$, under the $\mathrm{GL}(2, \mathbb{R})$-action discussed in §5.1, fixing α_1 in this way does not change the resulting set of SL 3-folds.

Theorem 10.1 *In the situation of §9, set $w_1(t) = \tanh t$ and $w_2(t) = w_3(t) =$*

$\operatorname{sech} t$*, as in case* (a) *of Theorem 9.3. Equation* (46) *then becomes*

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = \tfrac{1}{2}(\bar{p}_2 + \bar{p}_3)\operatorname{sech} t, \qquad \frac{\mathrm{d}p_2}{\mathrm{d}t} = -\tfrac{1}{2}(\bar{p}_1 \operatorname{sech} t + \bar{p}_3 \tanh t)$$
$$\text{and} \qquad \frac{\mathrm{d}p_3}{\mathrm{d}t} = -\tfrac{1}{2}(\bar{p}_1 \operatorname{sech} t + \bar{p}_2 \tanh t). \tag{52}$$

The full solutions to these equations are

$$p_1 = A \tanh t + B\bigl(f(t)\tanh t - 2\sqrt{\cosh t}\bigr) - \frac{iD\, f(t)}{\cosh t} + \frac{iE}{\sqrt{\cosh t}}, \tag{53}$$

$$\begin{aligned} p_2 = {}& A \operatorname{sech} t + B\, f(t)\operatorname{sech} t + C\sqrt{\cosh t} \\ &+ iD\Bigl(\cosh t - \frac{f(t)\sinh t}{2\sqrt{\cosh t}}\Bigr) + \frac{iE \sinh t}{2\sqrt{\cosh t}} + \frac{iF}{\sqrt{\cosh t}}, \end{aligned} \tag{54}$$

$$\begin{aligned} p_3 = {}& A \operatorname{sech} t + B\, f(t)\operatorname{sech} t - C\sqrt{\cosh t} \\ &+ iD\Bigl(\cosh t - \frac{f(t)\sinh t}{2\sqrt{\cosh t}}\Bigr) + \frac{iE \sinh t}{2\sqrt{\cosh t}} - \frac{iF}{\sqrt{\cosh t}}, \end{aligned} \tag{55}$$

where $A, B, C, D, E, F \in \mathbb{R}$ *and* $f(t) = \int_0^t \sqrt{\cosh s}\,\mathrm{d}s$.

Proof. Equation (52) follows immediately. One can verify that (53)–(55) are solutions to (52) by subtituting them in, and using $\frac{\mathrm{d}f}{\mathrm{d}t} = \sqrt{\cosh t}$. The six solutions of (52) this gives, with coefficients $A, \ldots, F$, are easily seen to be linearly independent. But as (52) is a well-behaved first-order o.d.e., its solutions are determined by the initial data $p_1(0), p_2(0), p_3(0)$, which comprise 3 complex or 6 real numbers. Thus (52) can have no more than 6 linearly independent solutions, so (53)–(55) are the full solutions to (52). □

To derive equations (53)–(55), the author used Lemma 9.5, the fact that $p_j = w_j$ is automatically a solution, and the symmetry between p_2 and p_3, which means that we can consider separately solutions with $p_2 = p_3$, and those with $p_1 = 0$ and $p_2 = -p_3$. Next we work out the conditions for (7)–(10) to hold for these solutions.

Lemma 10.2 *Let* w_1, w_2, w_3 *be as above, define* p_1, p_2, p_3 *by* (53)–(55) *for some* $A, \ldots, F \in \mathbb{R}$*, and define* q_1, q_2, q_3 *by* (53)–(55)*, but replacing* p_j *with* q_j *and* $A, \ldots, F$ *with* $A', \ldots, F' \in \mathbb{R}$*. This defines* $\mathbf{z}_1, \ldots, \mathbf{z}_5$*, as in §9. Equations* (7)–(10) *hold for these* $\mathbf{z}_1, \ldots, \mathbf{z}_5$ *if and only if*

$$D = D' = 0 \quad \text{and} \quad AD' + BE' + CF' = A'D + B'E + C'F. \tag{56}$$

Proof. When $t = 0$, we have $w_1(0) = 1$, $w_2(0) = w_3(0) = 1$, $p_1(0) = -2B + iE$, $p_2(0) = A + C + iD + iF$, $p_3(0) = A - C + iD - iF$, $q_1(0) = -2B' + iE'$, $q_2(0) = A' + C' + iD' + iF'$ and $q_3(0) = A' - C' + iD' - iF'$. Thus (49) holds at

$t = 0$ if and only if (56) holds, so by Lemma 9.4 equations (7)–(10) hold when $t = 0$ if and only if (56) holds. But (7)–(10) hold at $t = 0$ if and only if they hold for all t. $\square$

Thus we should set D and D' to zero. But we can show by changing coordinates in $\mathbb{R}^2$ from (y_1, y_2) to $(y_1 + A, y_2 + A')$ as in §5.1 that we are also free to set A and A' to zero, without restricting the SL 3-folds constructed in Theorem 5.1. So the remaining interesting parameters are B, C, E, F and B', C', E', F', which must satisfy the restriction $BE' + CF' = B'E + C'F$.

Drawing together much of the work above, we get the following result, which is the explicit working out of those special Lagrangian 3-folds of Theorem 5.1 coming out of part (a) of Theorem 9.3.

Theorem 10.3 *Define functions $w_j, p_j, q_j : \mathbb{R} \to \mathbb{C}$ for $j = 1, 2, 3$ by*

$$\begin{aligned}
w_1(t) &= \tanh t, \qquad w_2(t) = w_3(t) = \operatorname{sech} t, \\
p_1(t) &= B\bigl(f(t)\tanh t - 2\sqrt{\cosh t}\,\bigr) + \frac{iE}{\sqrt{\cosh t}}, \\
p_2(t) &= B\, f(t)\operatorname{sech} t + C\sqrt{\cosh t} + \frac{iE\,\sinh t}{2\sqrt{\cosh t}} + \frac{iF}{\sqrt{\cosh t}}, \\
p_3(t) &= B\, f(t)\operatorname{sech} t - C\sqrt{\cosh t} + \frac{iE\,\sinh t}{2\sqrt{\cosh t}} - \frac{iF}{\sqrt{\cosh t}}, \\
q_1(t) &= B'\bigl(f(t)\tanh t - 2\sqrt{\cosh t}\,\bigr) + \frac{iE'}{\sqrt{\cosh t}}, \\
q_2(t) &= B'\, f(t)\operatorname{sech} t + C'\sqrt{\cosh t} + \frac{iE'\,\sinh t}{2\sqrt{\cosh t}} + \frac{iF'}{\sqrt{\cosh t}}, \\
q_3(t) &= B'\, f(t)\operatorname{sech} t - C'\sqrt{\cosh t} + \frac{iE'\,\sinh t}{2\sqrt{\cosh t}} - \frac{iF'}{\sqrt{\cosh t}},
\end{aligned}$$

where $B, \ldots, F' \in \mathbb{R}$ satisfy $BE' + CF' = B'E + C'F$, and $f(t) = \int_0^t \sqrt{\cosh s}\,\mathrm{d}s$. Let $r_1, r_2, r_3 : \mathbb{R} \to \mathbb{C}$ be the unique solutions of

$$\frac{r_1}{\mathrm{d}t} = \tfrac{1}{2}(\bar p_2 \bar p_3 + \bar q_3 \bar q_2), \quad \frac{r_2}{\mathrm{d}t} = \tfrac{1}{2}(\bar q_3 \bar q_1 - \bar p_1 \bar p_3), \quad \frac{r_3}{\mathrm{d}t} = -\tfrac{1}{2}(\bar p_1 \bar q_2 + \bar p_2 \bar q_1)$$

with $r_j(0) = 0$. Define a subset N of $\mathbb{C}^3$ by

$$\begin{aligned}
N = \Bigl\{\bigl(&\tfrac{1}{2}(y_1^2 + y_2^2) w_1(t) + y_1 p_1(t) + y_2 q_1(t) + r_1(t), \\
&\tfrac{1}{2}(y_1^2 - y_2^2) w_2(t) + y_1 p_2(t) - y_2 q_2(t) + r_2(t), \\
&y_1 y_2 w_3(t) + y_1 q_3(t) + y_2 p_3(t) + r_3(t)\bigr) : t, y_1, y_2 \in \mathbb{R}\Bigr\}.
\end{aligned}$$

Then N is a special Lagrangian 3-fold.

Observe that if $E = E' = CF = C'F' = 0$ then w_1, p_1, q_1 and r_1 are real, so that $z_1 \in \mathbb{R}$ for each $(z_1, z_2, z_3) \in N$. This implies that writing $\mathbb{C}^3 = \mathbb{C} \times \mathbb{C}^2$,

we have $N \subseteq \mathbb{R} \times \Sigma$, where Σ is a special Lagrangian 2-fold in $\mathbb{C}^2$. This is not obvious from the explicit expression for N, because the coordinates t, y_1, y_2 are not compatible with the product structure on N. But SL 2-folds in $\mathbb{C}^2$ are holomorphic with respect to an alternative complex structure J on $\mathbb{C}^2$, and in fact Σ is a *complex parabola* with respect to J.

11 Case (d) of Theorem 9.3

Next we study the SL 3-folds N which come from the construction of Theorem 5.1 when $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ are as in case (d) of Theorem 9.3. Throughout this section we use the notation of §9. Let $\alpha_1, \alpha_2, \alpha_3 > 0$ satisfy $\frac{1}{\alpha_1} = \frac{1}{\alpha_2} + \frac{1}{\alpha_3}$, and as in case (d) of Theorem 9.3, define $a_1, a_2, a_3 \in \mathbb{R}$ by

$$\begin{gathered} a_1 = -\alpha_1^{-1/2}(\alpha_2\alpha_3)^{1/2}, \qquad a_2 = \alpha_2^{-1/2}(\alpha_3\alpha_1)^{1/2} \\ \text{and} \qquad a_3 = \alpha_3^{-1/2}(\alpha_1\alpha_2)^{1/2}, \end{gathered} \tag{57}$$

so that $a_1 + a_2 + a_3 = 0$. Define $w_1, w_2, w_3 : \mathbb{R} \to \mathbb{C}$ by

$$\begin{gathered} w_1(t) = i\sqrt{\alpha_1}\,\mathrm{e}^{ia_1 t}, \qquad w_2(t) = \sqrt{\alpha_2}\,\mathrm{e}^{ia_2 t} \\ \text{and} \qquad w_3(t) = \sqrt{\alpha_3}\,\mathrm{e}^{ia_3 t}. \end{gathered} \tag{58}$$

Then (41) holds, and when we define $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ by (40) they satisfy (13).

In §11.1 we solve equations (14) and (15) for $\mathbf{z}_4, \mathbf{z}_5$ and $\mathbf{z}_6$, and so write down in Theorem 11.5 a large family of explicit SL 3-folds in $\mathbb{C}^3$. Section 11.2 then studies solutions periodic in t, which are surprisingly abundant, interprets them geometrically, and gives a parameter count for the construction.

11.1 Solving the equations for $\mathbf{z}_4, \mathbf{z}_5$ and $\mathbf{z}_6$

We shall explicitly solve equations (14) and (15) for $\mathbf{z}_4, \mathbf{z}_5$ and $\mathbf{z}_6$ with this choice of $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$. But (14) is equivalent to equations (46) and (47) for p_1, p_2, p_3 and q_1, q_2, q_3, as in §9.2. As (46) and (47) are identical, we need only solve (46). Here is a convenient way of rewriting it.

Proposition 11.1 *In the situation above, write*

$$p_1 = i\mathrm{e}^{ia_1 t}\beta_1, \quad p_2 = \mathrm{e}^{ia_2 t}\beta_2 \quad \textit{and} \quad p_3 = \mathrm{e}^{ia_3 t}\beta_3, \tag{59}$$

where $\beta_j : \mathbb{R} \to \mathbb{C}$ is a differentiable function. Then (46) *is equivalent to*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \bar\beta_1 \\ \bar\beta_2 \\ \bar\beta_3 \end{pmatrix} = \tfrac{1}{2}i \begin{pmatrix} -2a_1 & 0 & 0 & 0 & -\sqrt{\alpha_3} & -\sqrt{\alpha_2} \\ 0 & -2a_2 & 0 & \sqrt{\alpha_3} & 0 & \sqrt{\alpha_1} \\ 0 & 0 & -2a_3 & \sqrt{\alpha_2} & \sqrt{\alpha_1} & 0 \\ 0 & \sqrt{\alpha_3} & \sqrt{\alpha_2} & 2a_1 & 0 & 0 \\ -\sqrt{\alpha_3} & 0 & -\sqrt{\alpha_1} & 0 & 2a_2 & 0 \\ -\sqrt{\alpha_2} & -\sqrt{\alpha_1} & 0 & 0 & 0 & 2a_3 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \bar\beta_1 \\ \bar\beta_2 \\ \bar\beta_3 \end{pmatrix}.$$

Proof. Substitute (59) into the first equation of (46), and use the values for w_1, w_2, w_3 in (58). We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(i\mathrm{e}^{ia_1 t}\beta_1\big) = \tfrac{1}{2}\mathrm{e}^{-i(a_2+a_3)t}(\sqrt{\alpha_3}\,\bar{\beta}_2 + \sqrt{\alpha_2}\,\bar{\beta}_3).$$

But $\mathrm{e}^{-i(a_2+a_3)t} = \mathrm{e}^{ia_1 t}$ as $a_1 + a_2 + a_3 = 0$. So dividing by $i\mathrm{e}^{ia_1 t}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta_1 + ia_1\beta_1 = -\tfrac{1}{2}i(\sqrt{\alpha_3}\,\bar{\beta}_2 + \sqrt{\alpha_2}\,\bar{\beta}_3),$$

which is equivalent to the first line of the equation we have to prove. The other lines follow in the same way. □

To solve for $\beta_1, \beta_2, \beta_3$ we need the eigenvalues and eigenvectors of this matrix.

Proposition 11.2 *Let M be the 6×6 real matrix appearing in Proposition 11.1. Then the eigenvalues of M are 0 (twice), $\lambda, -\lambda, 3\lambda$ and -3λ, where $\lambda > 0$ satisfies*

$$\lambda^2 = a_1^2 - a_2 a_3 = a_2^2 - a_3 a_1 = a_3^2 - a_1 a_2. \tag{60}$$

There exist in $\mathbb{R}^3$ nonzero real vectors (b_1, b_2, b_3), which is unique, (c_1, c_2, c_3), (d_1, d_2, d_3), (e_1, e_2, e_3) and (f_1, f_2, f_3), such that

$$M\begin{pmatrix}\sqrt{\alpha_1}\\ \sqrt{\alpha_2}\\ \sqrt{\alpha_3}\\ \sqrt{\alpha_1}\\ \sqrt{\alpha_2}\\ \sqrt{\alpha_3}\end{pmatrix} = 0, \qquad M\begin{pmatrix}b_1\\ b_2\\ b_3\\ -b_1\\ -b_2\\ -b_3\end{pmatrix} = \begin{pmatrix}\sqrt{\alpha_1}\\ \sqrt{\alpha_2}\\ \sqrt{\alpha_3}\\ \sqrt{\alpha_1}\\ \sqrt{\alpha_2}\\ \sqrt{\alpha_3}\end{pmatrix}, \tag{61}$$

$$M\begin{pmatrix}c_1\\ c_2\\ c_3\\ d_1\\ d_2\\ d_3\end{pmatrix} = \lambda\begin{pmatrix}c_1\\ c_2\\ c_3\\ d_1\\ d_2\\ d_3\end{pmatrix}, \qquad M\begin{pmatrix}d_1\\ d_2\\ d_3\\ c_1\\ c_2\\ c_3\end{pmatrix} = -\lambda\begin{pmatrix}d_1\\ d_2\\ d_3\\ c_1\\ c_2\\ c_3\end{pmatrix}, \tag{62}$$

$$M\begin{pmatrix}e_1\\ e_2\\ e_3\\ f_1\\ f_2\\ f_3\end{pmatrix} = 3\lambda\begin{pmatrix}e_1\\ e_2\\ e_3\\ f_1\\ f_2\\ f_3\end{pmatrix} \quad \textit{and} \quad M\begin{pmatrix}f_1\\ f_2\\ f_3\\ e_1\\ e_2\\ e_3\end{pmatrix} = -3\lambda\begin{pmatrix}f_1\\ f_2\\ f_3\\ e_1\\ e_2\\ e_3\end{pmatrix}. \tag{63}$$

Proof. From (57) we see that $M = LM'L^{-1}$, where

$$L = \begin{pmatrix} \sqrt{\alpha_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\alpha_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\alpha_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\alpha_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\alpha_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\alpha_3} \end{pmatrix}$$

$$\text{and} \qquad M' = \begin{pmatrix} -2a_1 & 0 & 0 & 0 & a_1 & a_1 \\ 0 & -2a_2 & 0 & a_2 & 0 & a_2 \\ 0 & 0 & -2a_3 & a_3 & a_3 & 0 \\ 0 & -a_1 & -a_1 & 2a_1 & 0 & 0 \\ -a_2 & 0 & -a_2 & 0 & 2a_2 & 0 \\ -a_3 & -a_3 & 0 & 0 & 0 & 2a_3 \end{pmatrix}.$$

Thus the eigenvalues of M are the same as those of M', which are the roots of the polynomial $p(x) = \det(M' - xI)$. Direct calculation shows that

$$\begin{aligned} p(x) = x^6 &+ \bigl(2(a_2a_3 + a_3a_1 + a_1a_2) - 4(a_1^2 + a_2^2 + a_3^2)\bigr)x^4 \\ &+ \bigl(9(a_2^2a_3^2 + a_3^2a_1^2 + a_1^2a_2^2) - 6(a_1^2a_2a_3 + a_2^2a_3a_1 + a_3^2a_1a_2)\bigr)x^2. \end{aligned}$$

Now $a_1 < 0$ and $a_2, a_3 > 0$ with $a_1 + a_2 + a_3 = 0$. Thus we may substitute $a_2 = -\sigma a_1$, $a_3 = (\sigma - 1)a_1$, for $\sigma = -a_2/a_1$ in $(0, 1)$. Then $p(x)$ becomes

$$p(x) = x^6 - 10a_1^2(\sigma^2 - \sigma - 1)x^4 + 9a_1^4(\sigma^4 - 2\sigma^3 + 3\sigma^2 - 2\sigma + 1)x^2,$$

which factorizes as

$$p(x) = x^2\bigl(x^2 - a_1^2(\sigma^2 - \sigma + 1)\bigr)\bigl(x^2 - 9a_1^2(\sigma^2 - \sigma + 1)\bigr).$$

Thus the roots of p are zero (twice), $\lambda, -\lambda, 3\lambda$ and -3λ, where

$$\lambda^2 = a_1^2(\sigma^2 - \sigma + 1) = a_1^2 - (-\sigma a_1)((\sigma - 1)a_1) = a_1^2 - a_2a_3.$$

This proves the first equation of (60). Note also that as $\sigma \in (0, 1)$ and $a_1 < 0$, we have $a_1^2(\sigma^2 - \sigma + 1) > 0$, so we can take λ to be real and positive. The second two equations of (60) follow from $a_1 + a_2 + a_3 = 0$, as for instance

$$(a_1^2 - a_2a_3) - (a_2^2 - a_3a_1) = (a_1 + a_2 + a_3)(a_1 - a_2) = 0.$$

The first equation of (61) follows from (57). The second is equivalent to

$$-\begin{pmatrix} 2a_1 & -\sqrt{\alpha_3} & -\sqrt{\alpha_2} \\ \sqrt{\alpha_3} & 2a_2 & \sqrt{\alpha_1} \\ \sqrt{\alpha_2} & \sqrt{\alpha_1} & 2a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \sqrt{\alpha_1} \\ \sqrt{\alpha_2} \\ \sqrt{\alpha_3} \end{pmatrix},$$

which has a unique solution (b_1, b_2, b_3), as the 3×3 matrix appearing here has determinant $4a_1a_2a_3 \neq 0$, and so is invertible. To prove (62), observe that

as λ is a real eigenvalue of the real matrix M, there exists a real eigenvector $(c_1\,c_2\,c_3\,d_1\,d_2\,d_3)^T$ of M with eigenvalue λ. This gives the first equation of (62), and the second then holds because of the form of M. One can also easily show that (c_1, c_2, c_3) and (d_1, d_2, d_3) are both nonzero. Equation (63) is proved in the same way, as 3λ is an eigenvalue of M. □

The following identities will be helpful later.

Proposition 11.3 *The constants $b_j, \ldots, f_j$ of Proposition 11.2 satisfy*

$$\sqrt{\alpha_1}\,c_1 - \sqrt{\alpha_2}\,c_2 - \sqrt{\alpha_3}\,c_3 - \sqrt{\alpha_1}\,d_1 + \sqrt{\alpha_2}\,d_2 + \sqrt{\alpha_3}\,d_3 = 0, \tag{64}$$
$$\sqrt{\alpha_1}\,e_1 - \sqrt{\alpha_2}\,e_2 - \sqrt{\alpha_3}\,e_3 - \sqrt{\alpha_1}\,f_1 + \sqrt{\alpha_2}\,f_2 + \sqrt{\alpha_3}\,f_3 = 0, \tag{65}$$
$$b_1c_1 - b_2c_2 - b_3c_3 + b_1d_1 - b_2d_2 - b_3d_3 = 0, \tag{66}$$
$$b_1e_1 - b_2e_2 - b_3e_3 + b_1f_1 - b_2f_2 - b_3f_3 = 0, \tag{67}$$
$$c_1e_1 - c_2e_2 - c_3e_3 - d_1f_1 + d_2f_2 + d_3f_3 = 0, \tag{68}$$
$$\textit{and} \qquad c_1f_1 - c_2f_2 - c_3f_3 - d_1e_1 + d_2e_2 + d_3e_3 = 0. \tag{69}$$

Furthermore, none of $\pm\lambda$ or $\pm 3\lambda$ is equal to a_1, a_2 or a_3.

Proof. The 6×6 matrix M studied above is not symmetric, because the signs of the $\sqrt{\alpha_1}$ terms are not right. However, if we define

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

then KMK^{-1} is a *symmetric* complex matrix, and Proposition 11.2 shows that

$$\begin{pmatrix} \sqrt{\alpha_1} \\ i\sqrt{\alpha_2} \\ i\sqrt{\alpha_3} \\ i\sqrt{\alpha_1} \\ \sqrt{\alpha_2} \\ \sqrt{\alpha_3} \end{pmatrix}, \quad \begin{pmatrix} c_1 \\ ic_2 \\ ic_3 \\ id_1 \\ d_2 \\ d_3 \end{pmatrix}, \quad \begin{pmatrix} d_1 \\ id_2 \\ id_3 \\ ic_1 \\ c_2 \\ c_3 \end{pmatrix}, \quad \begin{pmatrix} e_1 \\ ie_2 \\ ie_3 \\ if_1 \\ f_2 \\ f_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_1 \\ if_2 \\ if_3 \\ ie_1 \\ e_2 \\ e_3 \end{pmatrix}$$

are eigenvectors of KMK^{-1}, with eigenvalues $0, \lambda, -\lambda, 3\lambda$ and -3λ respectively.

Now if $\mathbf{u}, \mathbf{v}$ are eigenvectors of a symmetric matrix with different eigenvalues, then $\mathbf{u}^T\mathbf{v} = 0$. Taking inner products between the five eigenvectors above proves (64), (65), (68) and (69). Also, $(b_1, ib_2, ib_3, -ib_1, -b_2, -b_3)^T$ is essentially an eigenvector with eigenvalue 0, modulo nilpotent behaviour, and taking inner products of this with the λ and 3λ eigenvectors gives (66) and (67).

To prove the last part, as in the proof of Proposition 11.2 put $a_2 = -\sigma a_1$, $a_3 = (\sigma - 1)a_1$ and $\lambda^2 = a_1^2(\sigma^2 - \sigma + 1)$ for some $\sigma \in (0, 1)$. Then $a_1 = \pm\lambda$

when $a_1^2 = a_1^2(\sigma^2 - \sigma + 1)$, that is, when $\sigma = 0$ or 1, which contradicts $\sigma \in (0,1)$. Similarly, $a_2 = \pm\lambda$ when $a_1^2\sigma^2 = a_1^2(\sigma^2 - \sigma + 1)$, that is, when $\sigma = 1$, contradicting $\sigma \in (0,1)$, and $a_3 = \pm\lambda$ is ruled out in the same way. To eliminate $a_j = \pm 3\lambda$ for $j = 1,2,3$, note that $|a_j| \leqslant -a_1$, but $\lambda^2 = a_1^2(\sigma^2 - \sigma + 1)$ for $\sigma \in \mathbb{R}$ implies that $3\lambda \geqslant -3\sqrt{3}\,a_1/2 > -a_1$. □

From Proposition 11.2 we see that the o.d.e. in Proposition 11.1 has solution

$$\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \bar\beta_1\\ \bar\beta_2\\ \bar\beta_3\end{pmatrix} = (A - \tfrac{1}{2}Bt)\begin{pmatrix}\sqrt{\alpha_1}\\ \sqrt{\alpha_2}\\ \sqrt{\alpha_3}\\ \sqrt{\alpha_1}\\ \sqrt{\alpha_2}\\ \sqrt{\alpha_3}\end{pmatrix} + iB\begin{pmatrix}b_1\\ b_2\\ b_3\\ -b_1\\ -b_2\\ -b_3\end{pmatrix} + C\mathrm{e}^{i\lambda t/2}\begin{pmatrix}c_1\\ c_2\\ c_3\\ d_1\\ d_2\\ d_3\end{pmatrix} + \hat C\mathrm{e}^{-i\lambda t/2}\begin{pmatrix}d_1\\ d_2\\ d_3\\ c_1\\ c_2\\ c_3\end{pmatrix}$$
$$+ D\mathrm{e}^{3i\lambda t/2}\begin{pmatrix}e_1\\ e_2\\ e_3\\ f_1\\ f_2\\ f_3\end{pmatrix} + \hat D\mathrm{e}^{-3i\lambda t/2}\begin{pmatrix}f_1\\ f_2\\ f_3\\ e_1\\ e_2\\ e_3\end{pmatrix}$$

for $A, B, C, \hat C, D, \hat D \in \mathbb{C}$. But the last three rows of this equation should be the complex conjugates of the first three rows, which implies that A and B are real and $\hat C = \bar C$, $\hat D = \bar D$. This gives $\beta_1, \beta_2, \beta_3$, and then (59) gives the general solution for p_j. We have proved:

Proposition 11.4 *In the situation above, the general solution of* (46) *is*

$$\begin{aligned} p_1(t) = i\big((A - \tfrac{1}{2}Bt)\sqrt{\alpha_1}\,\mathrm{e}^{ia_1 t} + iBb_1\mathrm{e}^{ia_1 t} + Cc_1\mathrm{e}^{i(a_1+\lambda/2)t} \\ + \bar C d_1\mathrm{e}^{i(a_1-\lambda/2)t} + De_1\mathrm{e}^{i(a_1+3\lambda/2)t} + \bar D f_1\mathrm{e}^{i(a_1-3\lambda/2)t}\big),\end{aligned} \tag{70}$$

$$\begin{aligned} p_2(t) = (A - \tfrac{1}{2}Bt)\sqrt{\alpha_2}\,\mathrm{e}^{ia_2 t} + iBb_2\mathrm{e}^{ia_2 t} + Cc_2\mathrm{e}^{i(a_2+\lambda/2)t} \\ + \bar C d_2\mathrm{e}^{i(a_2-\lambda/2)t} + De_2\mathrm{e}^{i(a_2+3\lambda/2)t} + \bar D f_2\mathrm{e}^{i(a_2-3\lambda/2)t},\end{aligned} \tag{71}$$

$$\begin{aligned} p_3(t) = (A - \tfrac{1}{2}Bt)\sqrt{\alpha_3}\,\mathrm{e}^{ia_3 t} + iBb_3\mathrm{e}^{ia_3 t} + Cc_3\mathrm{e}^{i(a_3+\lambda/2)t} \\ + \bar C d_3\mathrm{e}^{i(a_3-\lambda/2)t} + De_3\mathrm{e}^{i(a_3+3\lambda/2)t} + \bar D f_3\mathrm{e}^{i(a_3-3\lambda/2)t},\end{aligned} \tag{72}$$

for $A, B \in \mathbb{R}$ and $C, D \in \mathbb{C}$, where $b_j, \ldots, f_j$ and λ are as in Proposition 11.2.

Define p_1, p_2, p_3 by (70)–(72), and similarly define q_1, q_2, q_3 by (70)–(72), but using constants $A', B' \in \mathbb{R}$ and $C', D' \in \mathbb{C}$ instead of A, B, C, D. Then p_j and q_j satisfy (46) and (47). From §9.2, if we define $\mathbf{z}_4, \mathbf{z}_5$ by (45), then they satisfy (14). We have found the general solution of the evolution equation (14) for $\mathbf{z}_4$ and $\mathbf{z}_5$, with the given values of $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$.

Now from §5, we need $\mathbf{z}_4$ and $\mathbf{z}_5$ to satisfy (8)–(10). By Lemma 9.4, these are equivalent to equation (49). Using the values of w_j and p_j above, we find

that

$$
\begin{aligned}
\operatorname{Im}(w_1\bar p_1 - w_2\bar p_2 - w_3\bar p_3) =& -B(\sqrt{\alpha_1}\,b_1 - \sqrt{\alpha_2}\,b_2 - \sqrt{\alpha_3}\,b_3)\\
& -\operatorname{Im}(C\mathrm{e}^{i\lambda t/2})(\sqrt{\alpha_1}\,c_1 - \sqrt{\alpha_2}\,c_2 - \sqrt{\alpha_3}\,c_3 - \sqrt{\alpha_1}\,d_1 + \sqrt{\alpha_2}\,d_2 + \sqrt{\alpha_3}\,d_3)\\
& -\operatorname{Im}(D\mathrm{e}^{3i\lambda t/2})(\sqrt{\alpha_1}\,e_1 - \sqrt{\alpha_2}\,e_2 - \sqrt{\alpha_3}\,e_3 - \sqrt{\alpha_1}\,f_1 + \sqrt{\alpha_2}\,f_2 + \sqrt{\alpha_3}\,f_3)\\
=& -B(\sqrt{\alpha_1}\,b_1 - \sqrt{\alpha_2}\,b_2 - \sqrt{\alpha_3}\,b_3),
\end{aligned}
$$

where in the last line we have used (64) and (65). Since in general $\sqrt{\alpha_1}\,b_1 - \sqrt{\alpha_2}\,b_2 - \sqrt{\alpha_3}\,b_3 \neq 0$, we see that $\operatorname{Im}(w_1\bar p_1 - w_2\bar p_2 - w_3\bar p_3) = 0$ if and only if $B = 0$.

Similarly, using (64)–(69) we find that $\operatorname{Im}(w_1\bar q_1 - w_2\bar q_2 - w_3\bar q_3) = 0$ if and only if $B' = 0$, and $\operatorname{Im}(p_1\bar q_1 - p_2\bar q_2 - p_3\bar q_3) = 0$ if and only if

$$
\begin{aligned}
&(A'B - AB')(\sqrt{\alpha_1}\,b_1 - \sqrt{\alpha_2}\,b_2 - \sqrt{\alpha_3}\,b_3) +\\
&\operatorname{Im}(C\bar C')(c_1^2 - c_2^2 - c_3^2 - d_1^2 + d_2^2 + d_3^2) + \operatorname{Im}(D\bar D')(e_1^2 - e_2^2 - e_3^2 - f_1^2 + f_2^2 + f_3^2) = 0.
\end{aligned}
$$

Thus we should set $B = B' = 0$. But we can show by changing coordinates in $\mathbb{R}^2$ from (y_1, y_2) to $(y_1 + A, y_2 + A')$ as in §5.1 that we are also free to set A and A' to be zero, without restricting the SL 3-folds constructed in Theorem 5.1.

Next we solve equation (15) for $\mathbf{z}_6$. From §9.2, this is equivalent to the o.d.e.s (48) for r_1, r_2, r_3, where $\mathbf{z}_6 = (r_1, r_2, r_3)$. Using the expressions above for p_j and q_j and remembering that $A = A' = B = B' = 0$, we get rather complicated expressions for $\mathrm{d}r_j/\mathrm{d}t$ for $j = 1, 2, 3$, which can then be integrated to get r_1, r_2 and r_3.

We sum up all the above work in the following result, which is the explicit working out of those special Lagrangian 3-folds of Theorem 5.1 coming out of part (d) of Theorem 9.3.

Theorem 11.5 *Define functions $w_j, p_j, q_j, r_j : \mathbb{R} \to \mathbb{C}$ for $j = 1, 2, 3$ by*

$$
\begin{aligned}
w_1(t) =& i\sqrt{\alpha_1}\,\mathrm{e}^{ia_1 t}, \quad w_2(t) = \sqrt{\alpha_2}\,\mathrm{e}^{ia_2 t}, \quad w_3(t) = \sqrt{\alpha_3}\,\mathrm{e}^{ia_3 t},\\
p_1(t) =& i\big(Cc_1\mathrm{e}^{i(a_1+\lambda/2)t} + \bar C d_1\mathrm{e}^{i(a_1-\lambda/2)t} + De_1\mathrm{e}^{i(a_1+3\lambda/2)t} + \bar D f_1\mathrm{e}^{i(a_1-3\lambda/2)t}\big),\\
p_2(t) =& Cc_2\mathrm{e}^{i(a_2+\lambda/2)t} + \bar C d_2\mathrm{e}^{i(a_2-\lambda/2)t} + De_2\mathrm{e}^{i(a_2+3\lambda/2)t} + \bar D f_2\mathrm{e}^{i(a_2-3\lambda/2)t},\\
p_3(t) =& Cc_3\mathrm{e}^{i(a_3+\lambda/2)t} + \bar C d_3\mathrm{e}^{i(a_3-\lambda/2)t} + De_3\mathrm{e}^{i(a_3+3\lambda/2)t} + \bar D f_3\mathrm{e}^{i(a_3-3\lambda/2)t},\\
q_1(t) =& i\big(C'c_1\mathrm{e}^{i(a_1+\lambda/2)t} + \bar C' d_1\mathrm{e}^{i(a_1-\lambda/2)t} + D'e_1\mathrm{e}^{i(a_1+3\lambda/2)t} + \bar D' f_1\mathrm{e}^{i(a_1-3\lambda/2)t}\big),\\
q_2(t) =& C'c_2\mathrm{e}^{i(a_2+\lambda/2)t} + \bar C' d_2\mathrm{e}^{i(a_2-\lambda/2)t} + D'e_2\mathrm{e}^{i(a_2+3\lambda/2)t} + \bar D' f_2\mathrm{e}^{i(a_2-3\lambda/2)t},\\
q_3(t) =& C'c_3\mathrm{e}^{i(a_3+\lambda/2)t} + \bar C' d_3\mathrm{e}^{i(a_3-\lambda/2)t} + D'e_3\mathrm{e}^{i(a_3+3\lambda/2)t} + \bar D' f_3\mathrm{e}^{i(a_3-3\lambda/2)t},\\
r_1(t) =& -\tfrac{i}{2(a_1+3\lambda)}(D^2 + D'^2)f_2 f_3\mathrm{e}^{i(a_1+3\lambda)t} - \tfrac{i}{2(a_1-3\lambda)}(\bar D^2 + \bar D'^2)e_2 e_3\mathrm{e}^{i(a_1-3\lambda)t}\\
& - \tfrac{i}{2(a_1+\lambda)}\big((C^2 + C'^2)d_2 d_3 + (\bar C D + \bar C' D')(c_2 f_3 + f_2 c_3)\big)\mathrm{e}^{i(a_1+\lambda)t}\\
& - \tfrac{i}{2(a_1-\lambda)}\big((\bar C^2 + \bar C'^2)c_2 c_3 + (C\bar D + C'\bar D')(d_2 e_3 + e_2 d_3)\big)\mathrm{e}^{i(a_1-\lambda)t} + E_1,
\end{aligned}
$$

$$
\begin{aligned}
r_2(t) = &\tfrac{1}{2(a_2+3\lambda)}(D^2 - D'^2) f_1 f_3 \mathrm{e}^{i(a_2+3\lambda)t} + \tfrac{1}{2(a_2-3\lambda)}(\bar{D}^2 - \bar{D}'^2) e_1 e_3 \mathrm{e}^{i(a_2-3\lambda)t} \\
&+ \tfrac{1}{2(a_2+\lambda)}\big((C^2 - C'^2) d_1 d_3 + (\bar{C}D - \bar{C}'D')(c_1 f_3 + f_1 c_3)\big)\mathrm{e}^{i(a_2+\lambda)t} \\
&+ \tfrac{1}{2(a_2-\lambda)}\big((\bar{C}^2 - \bar{C}'^2) c_1 c_3 + (C\bar{D} - C'\bar{D}')(d_1 e_3 + e_1 d_3)\big)\mathrm{e}^{i(a_2-\lambda)t} + E_2, \\
r_3(t) = &\tfrac{1}{a_3+3\lambda} DD' f_1 f_2 \mathrm{e}^{i(a_3+3\lambda)t} + \tfrac{1}{a_3-3\lambda}\bar{D}\bar{D}' \mathrm{e}^{i(a_3-3\lambda)t} \\
&+ \tfrac{1}{2(a_3+\lambda)}\big(2CC' d_1 d_2 + (\bar{C}D' + \bar{C}'D)(c_1 f_2 + f_1 c_2)\big)\mathrm{e}^{i(a_3+\lambda)t} \\
&+ \tfrac{1}{2(a_3-\lambda)}\big(2\bar{C}\bar{C}' c_1 c_2 + (C\bar{D}' + C'\bar{D})(d_1 e_2 + e_1 d_2)\big)\mathrm{e}^{i(a_3-\lambda)t} + E_3,
\end{aligned}
$$

where $E_1, E_2, E_3 \in \mathbb{C}$ *and* $C, D, C', D' \in \mathbb{C}$ *satisfy*

$$
\begin{aligned}
&\mathrm{Im}(C\bar{C}')(c_1^2 - c_2^2 - c_3^2 - d_1^2 + d_2^2 + d_3^2) \\
&+ \mathrm{Im}(D\bar{D}')(e_1^2 - e_2^2 - e_3^2 - f_1^2 + f_2^2 + f_3^2) = 0.
\end{aligned} \tag{73}
$$

Define a subset N *of* $\mathbb{C}^3$ *by*

$$
\begin{aligned}
N = \Big\{ \big(&\tfrac{1}{2}(y_1^2 + y_2^2) w_1(t) + y_1 p_1(t) + y_2 q_1(t) + r_1(t), \\
&\tfrac{1}{2}(y_1^2 - y_2^2) w_2(t) + y_1 p_2(t) - y_2 q_2(t) + r_2(t), \\
&y_1 y_2 w_3(t) + y_1 q_3(t) + y_2 p_3(t) + r_3(t)\big) : y_1, y_2, t \in \mathbb{R} \Big\}.
\end{aligned} \tag{74}
$$

Then N *is a special Lagrangian* 3*-fold.*

Note that in the expressions for r_1, r_2 and r_3 we have divided by factors $a_j \pm \lambda$ and $a_j \pm 3\lambda$ for $j = 1, 2, 3$. This is legitimate because none of $\pm\lambda$ or $\pm 3\lambda$ is equal to a_1, a_2 or a_3 by Proposition 11.3, so none of these factors vanish. If one of the factors had been zero, we would have had to replace the corresponding term by a multiple of t.

Observe that all the functions in the theorem are linear combinations of exponentials $\mathrm{e}^{i\alpha t}$ for $\alpha \in \mathbb{R}$. It would seem a reasonable guess that this is because the SL 3-fold N is actually symmetric under a subgroup U(1) or $\mathbb{R}$ in SU(3), which acts by multiplication by such exponentials in suitable coordinates. However, this is *not* the case, and generically the SL 3-folds of Theorem 11.5 have only discrete symmetry groups.

11.2 Periodicity conditions

We now discuss *periodicity* in t of the SL 3-folds N of Theorem 11.5. Let $\Phi : \mathbb{R}^3 \to \mathbb{C}^3$ be defined as in (24), so that $\Phi(y_1, y_2, t)$ is the vector in (74), and $N = \mathrm{Image}\,\Phi$. We want to know when $\Phi(y_1, y_2, t + T) = \Phi(y_1, y_2, t)$ for some $T > 0$ and all $y_1, y_2, t \in \mathbb{R}$. The corresponding immersed SL 3-folds N will then have topology $\mathcal{S}^1 \times \mathbb{R}^2$ rather than $\mathbb{R}^3$.

As in the proof of Proposition 11.2, write $a_2 = -\sigma a_1$, $a_3 = (\sigma - 1)a_1$ and $\lambda = -a_1\sqrt{\sigma^2 - \sigma + 1}$ for some $\sigma \in (0, 1)$. Also let $\tau = \sqrt{\sigma^2 - \sigma + 1}$, so that

$\tau > 0$ and $\lambda = -a_1\tau$. The exponentials in the expressions for w_j, p_j, q_j, r_j have the 27 periods

$$\frac{2\pi}{a_j}, \ \frac{2\pi}{a_j \pm \lambda/2}, \ \frac{2\pi}{a_j \pm \lambda}, \ \frac{2\pi}{a_j \pm 3\lambda/2} \ \text{ and } \ \frac{2\pi}{a_j \pm 3\lambda} \ \text{ for } j = 1, 2, 3. \tag{75}$$

For generic values of C, D, C', D', it is clear that Φ will be periodic if and only if these periods have a common multiple. But this holds exactly when σ and τ lie in $\mathbb{Q}$. So we need to understand the set of pairs of rational numbers (σ, τ) satisfying $\tau^2 = \sigma^2 - \sigma + 1$, with $\sigma \in (0, 1)$ and $\tau > 0$.

Now finding the rational points on a conic is a well-known problem in elementary number theory, and has a standard solution method, which is to parametrize the conic in the usual way. A suitable parametrization of the conic $\tau^2 = \sigma^2 - \sigma + 1$ is

$$\sigma = (1 - 2s)/(1 - s^2) \qquad \text{and} \qquad \tau = (1 - s + s^2)/(1 - s^2).$$

This has the property that σ and τ are both rational if and only if s is rational, and that $\sigma \in (0, 1)$ and $\tau > 0$ if and only if $s \in (0, \frac{1}{2})$.

For example, when $s = p/q$ for coprime $p, q \in \mathbb{Z}$ with $0 < 2p < q$, we may set

$$\begin{gathered} a_1 = p^2 - q^2, \quad a_2 = q^2 - 2pq, \quad a_3 = 2pq - p^2 \\ \text{and} \quad \lambda = p^2 - pq + q^2. \end{gathered} \tag{76}$$

Then a_1, a_2, a_3 and λ satisfy the equations above with this value of s. Careful consideration shows that

$$\mathrm{hcf}(a_1, a_2, a_3) = \mathrm{hcf}(a_1, a_2, a_3, \lambda) = \begin{cases} 1 & p + q \not\equiv 0 \mod 3, \\ 3 & p + q \equiv 0 \mod 3. \end{cases}$$

When $p + q \equiv 0 \mod 3$, we replace the a_j and λ by

$$\begin{gathered} a_1 = \tfrac{1}{3}(p^2 - q^2), \quad a_2 = \tfrac{1}{3}(q^2 - 2pq), \quad a_3 = \tfrac{1}{3}(2pq - p^2) \\ \text{and} \quad \lambda = \tfrac{1}{3}(p^2 - pq + q^2), \end{gathered} \tag{77}$$

so that in both cases we have $\mathrm{hcf}(a_1, a_2, a_3) = \mathrm{hcf}(a_1, a_2, a_3, \lambda) = 1$. Having chosen a_1, a_2, a_3, we determine $\alpha_1, \alpha_2, \alpha_3$ uniquely by inverting (57).

As p, q are coprime at least one of them is odd, so λ is odd. Thus $a_j, a_j \pm \lambda$ and $a_j \pm 3\lambda$ are integers, and $a_j \pm \lambda/2$ and $a_j \pm 3\lambda/2$ are half-integers but not integers. It follows that

$$w_j(t+2\pi) = w_j(t), \ \ p_j(t+2\pi) = -p_j(t), \ \ q_j(t+2\pi) = -q_j(t), \ \ r_j(t+2\pi) = r_j(t).$$

Hence $\Phi(y_1, y_2, t + 2\pi) = \Phi(-y_1, -y_2, t)$ for all $y_1, y_2, t \in \mathbb{R}$. This implies that $\Phi(y_1, y_2, t + 4\pi) = \Phi(y_1, y_2, t)$. Moreover, as $\mathrm{hcf}(a_1, a_2, a_3) = 1$ one can show that 4π is the least $T > 0$ with $\Phi(y_1, y_2, t + T) = \Phi(y_1, y_2, t)$ for all $y_1, y_2, t \in \mathbb{R}$, so that Φ is periodic in t with period 4π.

Therefore we can regard Φ as a map $\mathbb{R}^3/\mathbb{Z} \to \mathbb{C}^3$, where $\mathbb{Z}$ acts on $\mathbb{R}^3$ by

$$(y_1, y_2, y_3) \stackrel{n}{\longmapsto} \big((-1)^n y_1, (-1)^n y_2, t + 2\pi n\big) \qquad \text{for } n \in \mathbb{Z}. \tag{78}$$

As $\mathbb{R}^3/\mathbb{Z}$ is diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$, though not with the obvious $\mathbb{R}^2$ coordinates, we see that if Φ is an immersion then N is an immersed 3-submanifold diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$.

Consider the asymptotic behaviour of these 3-folds N at infinity in $\mathbb{C}^3$. From (74) we see that $\Phi(y_1, y_2, t) \to \infty$ in $\mathbb{C}^3$ as $(y_1, y_2) \to \infty$ in $\mathbb{R}^2$. But when (y_1, y_2) is large then the dominant terms in (74) are the quadratic terms in y_1 and y_2, and we can neglect the lower order terms. Thus we expect that N should be asymptotic to $N_0 = \Phi_0(\mathbb{R}^3)$ at infinity in $\mathbb{C}^3$, to leading order, where $\Phi_0 : \mathbb{R}^3 \to \mathbb{C}^3$ is given by

$$\Phi_0(y_1, y_2, t) = \big(\tfrac{i}{2}(y_1^2 + y_2^2)\sqrt{\alpha_1}\,\mathrm{e}^{ia_1 t}, \tfrac{1}{2}(y_1^2 - y_2^2)\sqrt{\alpha_2}\,\mathrm{e}^{ia_2 t}, y_1 y_2 \sqrt{\alpha_3}\,\mathrm{e}^{ia_3 t}\big).$$

Calculation shows that N_0 is an SL T^2-cone in $\mathbb{C}^3$, which may be written

$$\big\{(i\mathrm{e}^{ia_1 t}x_1, \mathrm{e}^{ia_2 t}x_2, \mathrm{e}^{ia_3 t}x_3) : x_j, t \in \mathbb{R}, \quad x_1 \geqslant 0, \quad a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = 0\big\}.$$

We have already met this example in [4], in particular in [4, §7, case (a)], [4, Th. 8.7] and [4, Ex. 9.5].

Regarding Φ_0 as mapping $\mathbb{R}^3/\mathbb{Z} \to N_0$, where $\mathbb{Z}$ acts on $\mathbb{R}^3$ by (78), we find that Φ_0 is generically 2:1, since $\Phi_0(y_1, y_2, t) = \Phi_0(-y_1, -y_2, t)$. So we should think of N as converging at infinity to a *double cover* of the T^2-cone N_0. That is, towards infinity two points of N converge to each point of N_0, and at infinity N is a T^2-cone which is wrapped twice round the T^2-cone N_0.

Note that the convergence of N to N_0 is of a rather weak kind. Let r be the radius function on $\mathbb{C}^3$, so that $y_1^2 + y_2^2 = O(r)$. The largest terms in (74) we have neglected are linear in y_1, y_2, and so they are $O(r^{1/2})$. Thus N 'converges' to N_0 to order $O(r^{1/2})$ for large r, so that N actually gets further away from N_0 towards infinity, rather than closer.

We summarize the material above in the following theorem.

Theorem 11.6 *For each $s \in (0, \frac{1}{2}) \cap \mathbb{Q}$ the construction of Theorem 11.5 yields a family of closed special Lagrangian 3-folds in $\mathbb{C}^3$ depending on E_1, E_2, E_3 and $C, D, C', D' \in \mathbb{C}$ satisfying (73). Generic members of the family are nonsingular immersed 3-submanifolds diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$. Each 3-fold is weakly asymptotic to order $O(r^{1/2})$ at infinity in $\mathbb{C}^3$ to a double cover of the special Lagrangian T^2-cone*

$$\big\{(i\mathrm{e}^{ia_1 t}x_1, \mathrm{e}^{ia_2 t}x_2, \mathrm{e}^{ia_3 t}x_3) : x_j, t \in \mathbb{R}, \quad x_1 \geqslant 0, \quad a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = 0\big\},$$

where $a_1, a_2, a_3 \in \mathbb{Z}$ depend on $s = p/q$ as in (76) or (77).

It seems rather odd to the author that the periodicity conditions in this problem turn out to have such a neat, and geometrically interesting, answer. We saw in (75) that for a general member of the family to be periodic we need 27 periods to be relatively rational. As these periods depend only on a_1, a_2 and a_3, one would expect this to be a very overdetermined problem, with no interesting solutions, but in fact there are infinitely many.

It is also surprising that once the rationality conditions are solved, when a_1, a_2, a_3 are integers it turns out that λ is necessarily an integer, rather than just a rational number. This has the effect that at infinity N is a double cover of N_0, rather than a multiple cover of some high degree.

We finish by giving a parameter count for the SL 3-folds of Theorem 11.5. They depend upon parameters $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and $C, D, C', D', E_1, E_2, E_3 \in \mathbb{C}$, which is 17 real parameters. These must satisfy $1/\alpha_1 = 1/\alpha_2 + 1/\alpha_3$ and (73), reducing it to 15 parameters. Of the symmetry groups $\mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ and $\mathrm{SU}(3) \ltimes \mathbb{C}^3$ we have used all but the dilations in $\mathrm{GL}(2, \mathbb{R})$ and the translations in $\mathbb{C}^3$. We must subtract 7 parameters for these, leaving 8 parameters.

There is one other symmetry to take account of, which is translation in time, $t \mapsto t + c$. This has the following effect: if we replace C, D, C', D' by $\mathrm{e}^{i\lambda c/2}C$, $\mathrm{e}^{3i\lambda c/2}D$, $\mathrm{e}^{i\lambda c/2}C'$ and $\mathrm{e}^{3i\lambda c/2}D'$ respectively, then the corresponding SL 3-fold N' is equivalent to N under an $\mathrm{SU}(3) \ltimes \mathbb{C}^3$ transformation. So subtracting one parameter, we see that the family of SL 3-folds from Theorem 11.5 up to automorphisms of $\mathbb{C}^3$ has dimension 7. For comparison, the whole family from Theorem 5.1 has dimension 9.

12 The family of SL 3-folds from Example 4.4

We now apply the 'evolution equation' construction of §3 to the set of affine evolution data defined in Example 4.4. The material of this section runs parallel to sections 5 and 8, and so we will leave out many of the details.

12.1 Application of the method of §3

As in Example 4.4, let $k \geqslant 1$, and let $(x_1, \dots, x_k, y_1, y_2)$ be coordinates on $\mathbb{R}^{k+2}$. Define P to be the image in $\mathbb{R}^{k+2}$ of the map $\psi : \mathbb{R}^2 \to \mathbb{R}^{k+2}$ given by

$$\psi : (x, y) \longmapsto (x, x^2, \dots, x^k, y, xy), \tag{79}$$

and $\chi : \mathbb{R}^{k+2} \to \Lambda^2 \mathbb{R}^{k+2}$ to be the affine map

$$\begin{aligned} &\chi(x_1, \dots, x_k, y_1, y_2) = \\ &\quad -2y_1 \tfrac{\partial}{\partial y_1} \wedge \tfrac{\partial}{\partial y_2} + 2 \tfrac{\partial}{\partial x_1} \wedge \tfrac{\partial}{\partial y_1} + 4x_1 \tfrac{\partial}{\partial x_2} \wedge \tfrac{\partial}{\partial y_1} + \cdots + 2k x_{k-1} \tfrac{\partial}{\partial x_k} \wedge \tfrac{\partial}{\partial y_1} \\ &\quad + 2x_1 \tfrac{\partial}{\partial x_1} \wedge \tfrac{\partial}{\partial y_2} + 4x_2 \tfrac{\partial}{\partial x_2} \wedge \tfrac{\partial}{\partial y_2} + \cdots + 2k x_k \tfrac{\partial}{\partial x_k} \wedge \tfrac{\partial}{\partial y_2}. \end{aligned} \tag{80}$$

Then (P, χ) is a set of affine evolution data with $m = 3$ and $n = k + 2$.

Let $\mathbf{p}_0, \ldots, \mathbf{p}_k$ and $\mathbf{q}_1, \mathbf{q}_2$ be vectors in $\mathbb{C}^3$, and define an affine map $\phi : \mathbb{R}^{k+2} \to \mathbb{C}^3$ by

$$\phi : (x_1, \ldots, x_k, y_1, y_2) \mapsto \mathbf{p}_0 + x_1\mathbf{p}_1 + \cdots + x_k\mathbf{p}_k + y_1\mathbf{q}_1 + y_2\mathbf{q}_2. \tag{81}$$

Then from (80) we see that

$$\begin{aligned}\phi^*(\omega) \cdot \chi = 2\bigl(&-y_1\omega(\mathbf{q}_1, \mathbf{q}_2) + \omega(\mathbf{p}_1, \mathbf{q}_1) + 2x_1\omega(\mathbf{p}_2, \mathbf{q}_1) + \cdots + kx_{k-1}\omega(\mathbf{p}_k, \mathbf{q}_1) \\ &+ x_1\omega(\mathbf{p}_1, \mathbf{q}_2) + 2x_2\omega(\mathbf{p}_2, \mathbf{q}_2) + \cdots + kx_k\omega(\mathbf{p}_k, \mathbf{q}_2)\bigr).\end{aligned}$$

Thus $\phi^*(\omega)|_P \equiv 0$ if and only if

$$\omega(\mathbf{q}_1, \mathbf{q}_2) = 0, \qquad \omega(\mathbf{p}_k, \mathbf{q}_2) = 0 \qquad \text{and} \tag{82}$$

$$j\,\omega(\mathbf{p}_j, \mathbf{q}_1) + (j-1)\omega(\mathbf{p}_{j-1}, \mathbf{q}_2) = 0 \quad \text{for } 1 \leqslant j \leqslant k. \tag{83}$$

These are the analogues of equations (7)–(10).

Let $\mathbf{p}_0(t), \ldots, \mathbf{p}_k(t)$ and $\mathbf{q}_1(t), \mathbf{q}_2(t)$ be differentiable functions $\mathbb{R} \to \mathbb{C}^3$, and define ϕ_t by (81) for $t \in \mathbb{R}$. Then as in §5, we use (80) to show that equation (1) of §3 holds for the family $\bigl\{\phi_t : t \in \mathbb{R}\bigr\}$ if and only if

$$\begin{aligned}\frac{\mathrm{d}\phi_t}{\mathrm{d}t}(x_1, \ldots, y_2) = &-2y_1\,\mathbf{q}_1 \times \mathbf{q}_2 + 2\,\mathbf{p}_1 \times \mathbf{q}_1 + 4x_1\,\mathbf{p}_2 \times \mathbf{q}_1 + \cdots + 2kx_{k-1}\,\mathbf{p}_k \times \mathbf{q}_1 \\ &+ 2x_1\,\mathbf{p}_1 \times \mathbf{q}_2 + 4x_2\,\mathbf{p}_2 \times \mathbf{q}_2 + \cdots + 2k\,x_k\mathbf{p}_k \times \mathbf{q}_2,\end{aligned}$$

where the cross product '$\times$' is defined in (12). Using (81) we get expressions for $\mathrm{d}\mathbf{p}_i/\mathrm{d}t$ and $\mathrm{d}\mathbf{q}_j/\mathrm{d}t$. So applying Theorem 3.2, we prove the following analogue of Theorem 5.1:

Theorem 12.1 *Let $k \geqslant 1$, and suppose $\mathbf{p}_0, \ldots, \mathbf{p_k}, \mathbf{q}_1, \mathbf{q}_2 : \mathbb{R} \to \mathbb{C}^3$ are differentiable functions satisfying equations* (82) *and* (83) *at $t = 0$ and*

$$\frac{\mathrm{d}\mathbf{p}_0}{\mathrm{d}t} = 2\,\mathbf{p}_1 \times \mathbf{q}_1, \qquad \frac{\mathrm{d}\mathbf{p}_k}{\mathrm{d}t} = 2k\,\mathbf{p}_k \times \mathbf{q}_2, \tag{84}$$

$$\frac{\mathrm{d}\mathbf{p}_j}{\mathrm{d}t} = 2(j+1)\,\mathbf{p}_{j+1} \times \mathbf{q}_1 + 2j\,\mathbf{p}_j \times \mathbf{q}_2 \quad \textit{for } 1 \leqslant j \leqslant k-1, \tag{85}$$

$$\frac{\mathrm{d}\mathbf{q}_1}{\mathrm{d}t} = -2\,\mathbf{q}_1 \times \mathbf{q}_2 \qquad \textit{and} \qquad \frac{\mathrm{d}\mathbf{q}_2}{\mathrm{d}t} = 0 \tag{86}$$

for all $t \in \mathbb{R}$, where '$\times$' is as in (12)*. Define a subset N of $\mathbb{C}^3$ to be*

$$\bigl\{\mathbf{p}_0(t) + x\,\mathbf{p}_1(t) + \cdots + x^k\,\mathbf{p}_k(t) + y\,\mathbf{q}_1(t) + xy\,\mathbf{q}_2(t) : x, y, t \in \mathbb{R}\bigr\}. \tag{87}$$

Then N is a special Lagrangian 3-fold in $\mathbb{C}^3$ wherever it is nonsingular.

As in §5, if (82) and (83) hold at $t = 0$ then they hold for all $t \in \mathbb{R}$, and given initial values $\mathbf{p}_j(0), \mathbf{q}_j(0)$, there exist unique solutions $\mathbf{p}_j(t), \mathbf{q}_k(t)$ to (84)–(86) for t in $(-\epsilon, \epsilon)$ and some small $\epsilon > 0$. In fact solutions always exist for all $t \in \mathbb{R}$, and this is why we have used $t \in \mathbb{R}$ rather than $t \in (-\epsilon, \epsilon)$ above.

Following (24), define $\Phi : \mathbb{R}^3 \to \mathbb{C}^3$ by

$$\Phi(x,y,t) = \mathbf{p}_0(t) + x\,\mathbf{p}_1(t) + \cdots + x^k\,\mathbf{p}_k(t) + y\,\mathbf{q}_1(t) + xy\,\mathbf{q}_2(t), \tag{88}$$

so that $N = \mathrm{Image}\,\Phi$. As in §6, one can show that Φ fails to be an immersion for a subset of real codimension one in the set of all initial data $\mathbf{p}_i(0), \mathbf{q}_j(0)$ satisfying (82) and (83). In particular, for *generic* initial data Φ is an immersion, and N a nonsingular immersed 3-submanifold diffeomorphic to $\mathbb{R}^3$.

Observe also that the SL 3-folds of Theorem 12.1 are ruled by straight lines, as we showed in §8.3 for the 3-folds of Theorem 8.4. As (88) contains no terms in y^2, for each fixed $x, t \in \mathbb{R}$ the set $\{\Phi(x,y,t) : y \in \mathbb{R}\}$ is a real straight line in $\mathbb{C}^3$. So N is fibred by straight lines, and is a *ruled submanifold*. Ruled SL 3-folds are studied in [7].

We have already met the families of SL 3-folds constructed above when $k = 1$ and 2. When $k = 1$ we have

$$\Phi(x,y,t) = \mathbf{p}_0(t) + x\,\mathbf{p}_1(t) + y\,\mathbf{q}_1(t) + xy\,\mathbf{q}_2(t).$$

Clearly, $N = \mathrm{Image}\,\Phi$ is fibred by the images of the quadric $x_3 = x_1x_2$ in $\mathbb{R}^3$ under the affine maps $\phi_t : \mathbb{R}^3 \to \mathbb{C}^3$ given by

$$\phi_t : (x_1, x_2, x_3) \mapsto \mathbf{p}_0(t) + x_1\,\mathbf{p}_1(t) + x_2\,\mathbf{q}_1(t) + x_3\,\mathbf{q}_2(t).$$

It is easy to show that these SL 3-folds are isomorphic to those constructed in [5, Ex. 7.5], by evolving the image of an equivalent quadric in $\mathbb{R}^3$.

When $k = 2$, the SL 3-folds above are equivalent to those of §8, under automorphisms of $\mathbb{C}^3$. Putting $k = 2$ in (88) gives

$$\Phi(x,y,t) = \mathbf{p}_0(t) + x\,\mathbf{p}_1(t) + x^2\,\mathbf{p}_2(t) + y\,\mathbf{q}_1(t) + xy\,\mathbf{q}_2(t),$$

whereas in the construction of §8 we have

$$\Phi(y_1,y_2,t) = y_1^2\,\mathbf{z}_1(t) + y_1y_2\,\mathbf{z}_3(t) + y_1\,\mathbf{z}_4(t) + y_2\mathbf{z}_5(t) + \mathbf{z}_6(t),$$

remembering that $\mathbf{z}_1 \equiv \mathbf{z}_2$. Comparing these two equations, we see that they agree under the correspondence

$$x \leftrightarrow y_1, \quad y \leftrightarrow y_2, \quad \mathbf{p}_0 \leftrightarrow \mathbf{z}_6, \quad \mathbf{p}_1 \leftrightarrow \mathbf{z}_4, \quad \mathbf{p}_2 \leftrightarrow \mathbf{z}_1, \quad \mathbf{q}_1 \leftrightarrow \mathbf{z}_5, \quad \mathbf{q}_2 \leftrightarrow \mathbf{z}_3.$$

Also, the variable t in §8 corresponds to $2t$ in this section, which is due to the fact that in Example 4.2 we constructed χ from $\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}$, but in Example 4.4 we constructed χ from $2\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$.

12.2 Symmetries of the construction

In §5.1 we constructed an action of $\mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ on the set of solutions $\mathbf{z}_j$ to (13)–(15), which acts trivially on the corresponding SL 3-folds N of (16). This

group $\mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ acts on $\mathbb{R}^2$, and consists of the original symmetry group $G = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ used to construct the evolution data in Example 4.2, together with dilations of $\mathbb{R}^2$.

We shall now do the same thing for the construction above. In this case, the appropriate group of symmetries of $\mathbb{R}^2$ is the set of transformations of the form

$$(x, y) \mapsto (ax + b, cy + d_0 + d_1 x + \cdots + d_{k-1} x^{k-1}), \tag{89}$$

where a, b, c and $d_0, \ldots, d_{k-1} \in \mathbb{R}$, and $\delta = ac$ is nonzero. The subgroup of transformations with $\delta = 1$ and $a, c > 0$ are the group $G \ltimes U_k$ used in §4 to construct Example 4.4.

Here $G = \mathbb{R}_+ \ltimes \mathbb{R}$ is the a, b and $c = a^{-1}$ part of the action, and $U_k \cong \mathbb{R}^k$ is the vector space of 1-forms $p(x)\mathrm{d}y$ for $p(x)$ a polynomial of degree less than k, so that $p(x) = d_0 + d_1 x + \cdots + d_{k-1} x^{k-1}$. By allowing δ to be nonzero rather than fixing it to be 1 we include dilations of $\mathbb{R}^2$ in the group.

Following the proof of Proposition 5.2, we may show that the transformation (89) of $\mathbb{R}^2$ corresponds to the following transformation of the $\mathbf{p}_i$ and $\mathbf{q}_j$.

Proposition 12.2 *Suppose* $\mathbf{p}_0, \ldots, \mathbf{p}_k, \mathbf{q}_1, \mathbf{q}_2 : \mathbb{R} \to \mathbb{C}^3$ *satisfy* (84)–(86). *Let* a, b, c *and* $d_0, \ldots, d_{k-1} \in \mathbb{R}$ *with* $\delta = ac \neq 0$, *and define* $\mathbf{p}'_0, \ldots, \mathbf{p}'_k, \mathbf{q}'_1, \mathbf{q}'_2 : \mathbb{R} \to \mathbb{C}^3$ *by*

$$\begin{aligned}
&\mathbf{p}'_0(t) = \sum_{i=0}^{k} b^i\, \mathbf{p}_i(\delta t) + d_0\, \mathbf{q}_1(\delta t) + b d_0\, \mathbf{q}_2(\delta t),\\
&\mathbf{p}'_j(t) = \sum_{i=j}^{k} \binom{i}{j} a^j b^{i-j}\, \mathbf{p}_i(\delta t) + d_j\, \mathbf{q}_1(\delta t) + (a d_{j-1} + b d_j)\, \mathbf{q}_2(\delta t), \quad 1 \leqslant j < k,\\
&\mathbf{p}'_k(t) = a^k\, \mathbf{p}_k(\delta t) + a d_{k-1}\, \mathbf{q}_2(\delta t),\\
&\mathbf{q}'_1(t) = c\, \mathbf{q}_1(\delta t) + bc\, \mathbf{q}_2(\delta t) \qquad \textit{and} \qquad \mathbf{q}'_2(t) = ac\, \mathbf{q}_2(\delta t).
\end{aligned}$$

Then $\mathbf{p}'_0, \ldots, \mathbf{p}'_k, \mathbf{q}'_1, \mathbf{q}'_2$ *satisfy* (84)–(86). *Furthermore, the* $\mathbf{p}'_i, \mathbf{q}'_j$ *satisfy* (82) *and* (83) *if and only if the* $\mathbf{p}_i, \mathbf{q}_j$ *do, and then the special Lagrangian* 3*-folds* N, N' *constructed in* (87) *from the* $\mathbf{p}_j, \mathbf{q}_j$ *and* $\mathbf{p}'_i, \mathbf{q}'_j$ *are the same.*

We can now give a parameter count for the construction, following the method of §5.2. The initial data $\mathbf{p}_0(0), \ldots, \mathbf{p}_k(0), \mathbf{q}_1(0), \mathbf{q}_2(0)$ has $6(k+3)$ parameters. These must satisfy (82) and (83), which is $k+2$ equations. Thus the set $\mathcal{C}_P$ of Definition 3.1 has dimension $5k + 16$.

From this we must subtract symmetries of three kinds. Firstly, the internal symmetry group of Proposition 12.2 has dimension $k+3$. Secondly, the automorphisms $\mathrm{SU}(3) \ltimes \mathbb{C}^3$ of $\mathbb{C}^3$ have dimension 14. Thirdly, we subtract one dimension to allow for translation in time, $t \mapsto t + c$. Taking all these into account, we find that the family of distinct special Lagrangian 3-folds in $\mathbb{C}^3$ constructed in Theorem 12.1, up to automorphisms of $\mathbb{C}^3$, has dimension $4k - 2$.

12.3 Solving the equations

We shall now solve equations (84)–(86) for the $\mathbf{p}_i$ and $\mathbf{q}_j$ fairly explicitly, under the restrictions (82) and (83), and making use of the symmetries discussed above. Our treatment follows §8, and indeed the case $k = 2$ is equivalent to the construction of §8.

We begin by putting $\mathbf{q}_1, \mathbf{q}_2$ in a convenient form. Divide into the two cases

(a) $\mathbf{q}_1(0)$ and $\mathbf{q}_2(0)$ are linearly dependent, and

(b) $\mathbf{q}_1(0)$ and $\mathbf{q}_2(0)$ are linearly independent.

It is easy to show from (86) that in case (a) $\mathbf{q}_1, \mathbf{q}_2$ are constant, and the SL 3-fold N of (87) splits as a product $\Sigma \times \mathbb{R}$ in $\mathbb{C}^2 \times \mathbb{C}$, where Σ is an SL 2-fold in $\mathbb{C}^2$. Thus case (a) is not very interesting, and we shall not consider it further.

In case (b) we may follow the proof of Theorem 8.2 to prove:

Proposition 12.3 *Let $\mathbf{p}_0, \ldots, \mathbf{p}_k$ and $\mathbf{q}_1, \mathbf{q}_2$ satisfy equations (82)–(86), and suppose $\mathbf{q}_1(0), \mathbf{q}_2(0)$ are linearly independent. Then we may transform the $\mathbf{p}_i, \mathbf{q}_j$ under $\mathrm{SU}(3)$ and the symmetries of Proposition 12.2 to $\mathbf{p}'_i, \mathbf{q}'_j$, where*

$$\mathbf{q}'_1(t) = (\mathrm{e}^{it}, i\mathrm{e}^{-it}, 0) \quad \text{and} \quad \mathbf{q}'_2(t) = (0, 0, 1).$$

So let us fix $\mathbf{q}_1(t) = (\mathrm{e}^{it}, i\mathrm{e}^{-it}, 0)$ and $\mathbf{q}_2(t) = (0, 0, 1)$ for the rest of the section. Next we rewrite equations (82)–(85) for $\mathbf{p}_0, \ldots, \mathbf{p}_k$. The proof follows immediately from (12) and the definition of ω.

Proposition 12.4 *Set $\mathbf{q}_1(t) = (\mathrm{e}^{it}, i\mathrm{e}^{-it}, 0)$ and $\mathbf{q}_2(t) = (0, 0, 1)$, and write $\mathbf{p}_j = (a_j, b_j, c_j)$ for $j = 0, \ldots, k$, where $a_j, b_j, c_j : \mathbb{R} \to \mathbb{C}$ are differentiable functions. Then (86) holds, and equations (84) and (85) are equivalent to*

$$\frac{\mathrm{d}a_0}{\mathrm{d}t} = i\mathrm{e}^{it}\bar{c}_1, \quad \frac{\mathrm{d}b_0}{\mathrm{d}t} = \mathrm{e}^{-it}\bar{c}_1, \quad \frac{\mathrm{d}c_0}{\mathrm{d}t} = -i\mathrm{e}^{it}\bar{a}_1 - \mathrm{e}^{-it}\bar{b}_1, \tag{90}$$

$$\frac{\mathrm{d}a_j}{\mathrm{d}t} = (j+1)i\mathrm{e}^{it}\bar{c}_{j+1} + j\bar{b}_j \qquad \text{for } j = 1, \ldots, k-1, \tag{91}$$

$$\frac{\mathrm{d}b_j}{\mathrm{d}t} = (j+1)\mathrm{e}^{-it}\bar{c}_{j+1} - j\bar{a}_j \qquad \text{for } j = 1, \ldots, k-1, \tag{92}$$

$$\frac{\mathrm{d}c_j}{\mathrm{d}t} = -(j+1)(i\mathrm{e}^{it}\bar{a}_{j+1} + \mathrm{e}^{-it}\bar{b}_{j+1}) \quad \text{for } j = 1, \ldots, k-1, \tag{93}$$

$$\frac{\mathrm{d}a_k}{\mathrm{d}t} = k\bar{b}_k, \qquad \frac{\mathrm{d}b_k}{\mathrm{d}t} = -k\bar{a}_k \qquad \text{and} \qquad \frac{\mathrm{d}c_k}{\mathrm{d}t} = 0. \tag{94}$$

Furthermore, equations (82) and (83) are equivalent to $\operatorname{Im} c_k = 0$ and

$$j \operatorname{Im}(\mathrm{e}^{-it}a_j - i\mathrm{e}^{it}b_j) + (j-1)\operatorname{Im} c_{j-1} = 0 \quad \text{for } j = 1, \ldots, k. \tag{95}$$

The best way to solve equations (90)–(94) is in reverse order. That is, we begin by solving (94) for a_k, b_k, c_k. Then we inductively solve equations (91)–(93) for a_j, b_j, c_j with $j = k-1, k-2, \dots, 1$, treating a_{j+1}, b_{j+1} and c_{j+1} as known. Note that we can combine (91) and (92) to get

$$\frac{\mathrm{d}^2 a_j}{\mathrm{d}t^2} + j^2 a_j = j(j+1)\mathrm{e}^{it} c_{j+1} + (j+1)\frac{\mathrm{d}}{\mathrm{d}t}\big(i\mathrm{e}^{it}\bar{c}_{j+1}\big), \tag{96}$$

which is a linear second-order o.d.e. for a_j with prescribed right hand side, and can be solved by standard techniques. Finally we solve (90) for a_0, b_0, c_0, which is just a matter of integration. Here are the first four steps in this process.

The general solutions of (94) are easily shown to be

$$\begin{gathered} a_k(t) = A_k \mathrm{e}^{ikt} + B_k \mathrm{e}^{-ikt}, \qquad b_k(t) = i\bar{B}_k \mathrm{e}^{ikt} - i\bar{A}_k \mathrm{e}^{-ikt} \\ \text{and} \qquad c_k(t) = C_k, \end{gathered} \tag{97}$$

for constants $A_k, B_k, C_k \in \mathbb{C}$. The condition $\operatorname{Im} c_k = 0$ gives $\operatorname{Im} C_k = 0$. But applying Proposition 12.2 with $a = c = 1$, $b = d_0 = \cdots = d_{k-2} = 0$ and $d_{k-1} = -\operatorname{Re} C_k$ shows that we can use the symmetries of the construction to set $\operatorname{Re} C_k = 0$ as well. So let us set $C_k = 0$.

With these values for a_k, b_k, c_k we can solve equations (91)–(93) for a_{k-1}, b_{k-1} and c_{k-1}. We get

$$a_{k-1}(t) = A_{k-1}\mathrm{e}^{i(k-1)t} + B_{k-1}\mathrm{e}^{-i(k-1)t}, \tag{98}$$

$$b_{k-1}(t) = i\bar{B}_{k-1}\mathrm{e}^{i(k-1)t} - i\bar{A}_{k-1}\mathrm{e}^{-i(k-1)t}, \tag{99}$$

$$\begin{aligned} \text{and} \quad c_{k-1}(t) = &-\tfrac{k}{k-1}A_k \mathrm{e}^{i(k-1)t} + \tfrac{k}{k-1}\bar{A}_k \mathrm{e}^{-i(k-1)t} \\ &- \tfrac{k}{k+1}B_k \mathrm{e}^{-i(k+1)t} - \tfrac{k}{k+1}\bar{B}_k \mathrm{e}^{i(k+1)t} + C_{k-1}, \end{aligned} \tag{100}$$

for $A_{k-1}, B_{k-1}, C_{k-1} \in \mathbb{C}$, provided $k \neq 1$. When $k = 1$ the terms in $A_k, \bar{A}_k$ in (100) are replaced by $-2i\operatorname{Re}(A_1)t$. The case $j = k$ of (95) reduces to $\operatorname{Im} C_{k-1} = 0$, and as above we can use a symmetry involving the variable d_{k-2} in Proposition 12.2 to set $\operatorname{Re} C_{k-1} = 0$ too. So fix $C_{k-1} = 0$.

Repeating the same process for a_{k-2}, b_{k-2} and c_{k-2}, we obtain

$$\begin{aligned} a_{k-2}(t) = &A_{k-2}\mathrm{e}^{i(k-2)t} + B_{k-2}\mathrm{e}^{-i(k-2)t} \\ &+ \tfrac{k}{2}A_k\mathrm{e}^{ikt} + \tfrac{k(k-1)}{2(k+1)}B_k\mathrm{e}^{-ikt} - \tfrac{(k-1)}{2(k+1)}\bar{B}_k\mathrm{e}^{i(k+2)t}, \end{aligned} \tag{101}$$

$$\begin{aligned} b_{k-2}(t) = &i\big(\bar{B}_{k-2} - \tfrac{k}{k-2}A_k\big)\mathrm{e}^{i(k-2)t} - i\bar{A}_{k-2}\mathrm{e}^{-i(k-2)t} \\ &- \tfrac{i(k-1)}{2(k+1)}B_k\mathrm{e}^{-i(k+2)t} + \tfrac{ik(k-1)}{2(k+1)}\bar{B}_k\mathrm{e}^{ikt} - \tfrac{ik}{2}\bar{A}_k\mathrm{e}^{-ikt}, \end{aligned} \tag{102}$$

$$\begin{aligned} c_{k-2}(t) = &-\tfrac{k-1}{k-2}A_{k-1}\mathrm{e}^{i(k-2)t} + \tfrac{k-1}{k-2}\bar{A}_{k-1}\mathrm{e}^{-i(k-2)t} \\ &- \tfrac{k-1}{k}B_{k-1}\mathrm{e}^{-ikt} - \tfrac{k-1}{k}\bar{B}_{k-1}\mathrm{e}^{ikt} + C_{k-2}, \end{aligned} \tag{103}$$

for $A_{k-2}, B_{k-2}, C_{k-2} \in \mathbb{C}$, provided $k \neq 2$. When $k = 2$ the terms in $A_{k-1}, \bar{A}_{k-1}$ in (103) should be $-2i\operatorname{Re}(A_1)t$. Putting $j = k-1$ in (95) gives $\operatorname{Im} C_{k-2} = 0$, and we can again use symmetry to set $\operatorname{Re} C_{k-2} = 0$, so we fix $C_{k-2} = 0$.

There is no term in $\bar{A}_k$ in (101), because the $\bar{A}_k$ terms on the right hand side of (96) with $j = k-2$ cancel out. If this had not happened, then (101) and (102) would have included multiples of $t\mathrm{e}^{\pm i(k-2)t}$. This will be significant later, when we consider periodicity of the functions a_j, b_j, c_j.

Applying the same process for a_{k-3}, b_{k-3} and c_{k-3}, we obtain

$$\begin{aligned} a_{k-3}(t) = A_{k-3}\mathrm{e}^{i(k-3)t} + B_{k-3}\mathrm{e}^{-i(k-3)t} + \tfrac{k-1}{2}A_{k-1}\mathrm{e}^{i(k-1)t} \\ + \tfrac{(k-1)(k-2)}{2k}B_{k-1}\mathrm{e}^{-i(k-1)t} - \tfrac{(k-2)}{2k}\bar{B}_{k-1}\mathrm{e}^{i(k+1)t}, \end{aligned} \tag{104}$$

$$\begin{aligned} b_{k-3}(t) = i\big(\bar{B}_{k-3} - \tfrac{k-1}{k-3}A_{k-1}\big)\mathrm{e}^{i(k-3)t} - i\bar{A}_{k-3}\mathrm{e}^{-i(k-3)t} \\ -\tfrac{i(k-2)}{2k}B_{k-1}\mathrm{e}^{-i(k+1)t} + \tfrac{i(k-1)(k-2)}{2k}\bar{B}_{k-1}\mathrm{e}^{i(k-1)t} - \tfrac{i(k-1)}{2}\bar{A}_{k-1}\mathrm{e}^{-i(k-1)t}, \end{aligned} \tag{105}$$

$$\begin{aligned} c_{k-3}(t) = -\tfrac{k-2}{k-3}A_{k-2}\mathrm{e}^{i(k-3)t} + \tfrac{k-2}{k-3}\bar{A}_{k-2}\mathrm{e}^{-i(k-3)t} \\ -\tfrac{k-2}{k-1}\big(B_{k-2} - \tfrac{k^2}{2(k-2)}\bar{A}_k\big)\mathrm{e}^{-i(k-1)t} - \tfrac{k-2}{k-1}\big(\bar{B}_{k-2} + \tfrac{k}{2}A_k\big)\mathrm{e}^{i(k-1)t} \\ -\tfrac{(k-1)(k-2)}{2(k+1)}B_k\mathrm{e}^{-i(k+1)t} - \tfrac{(k-1)(k-2)}{2(k+1)}\bar{B}_k\mathrm{e}^{i(k+1)t} + C_{k-3}, \end{aligned} \tag{106}$$

for $A_{k-3}, B_{k-3}, C_{k-3} \in \mathbb{C}$, provided $k \neq 3$. When $k = 3$ the terms in $A_{k-2}, \bar{A}_{k-2}$ in (106) should be $-2i\,\mathrm{Re}(A_1)t$. As above we may fix $C_{k-3} = 0$.

The reader may readily carry on calculating a_j, b_j, c_j for $j = k-4, k-5, \ldots$ by this method. The expressions get steadily longer and more complicated, so we shall stop at this point. In Theorem 12.7 we will give the general form of the solutions a_j, b_j, c_j for all j, without the constant factors depending on j and k.

As an example, let $k = 4$. Then equations (97)–(106) give a_j, b_j, c_j for $j = 1, 2, 3, 4$ in terms of complex constants $A_1, \ldots, A_4$ and $B_1, \ldots, B_4$, and we get a_0, b_0, c_0 by integrating (90). Equation (95) holds for $j = 2, 3, 4$ by construction. However, we still have to satisfy (95) for $j = 1$. This is $\mathrm{Im}(\mathrm{e}^{-it}a_1 - i\mathrm{e}^{it}b_1) = 0$, which simplifies to $2\,\mathrm{Im}\,A_1 = 0$ after substituting in for a_1, b_1. So $A_1 \in \mathbb{R}$, and from Theorem 12.1 we deduce:

Theorem 12.5 *Define functions* $\mathbf{p}_0, \ldots, \mathbf{p}_4, \mathbf{q}_1, \mathbf{q}_2 : \mathbb{R} \to \mathbb{C}^3$ *by*

$$\begin{aligned} \mathbf{p}_0(t) = \big(& -2i\bar{A}_2 t + A_2\mathrm{e}^{2it} - \tfrac{1}{6}(\bar{B}_2 - 4A_4)\mathrm{e}^{4it} \\ & + \tfrac{1}{3}(B_2 + 2\bar{A}_4)\mathrm{e}^{-2it} - \tfrac{1}{10}\bar{B}_4\mathrm{e}^{6it} + \tfrac{3}{20}B_4\mathrm{e}^{-4it}, \\ & 2A_2 t - i\bar{A}_2\mathrm{e}^{-2it} + \tfrac{i}{3}(\bar{B}_2 - 4A_4)\mathrm{e}^{2it} \\ & - \tfrac{i}{6}(B_2 + 2\bar{A}_4)\mathrm{e}^{-4it} - \tfrac{i}{10}B_4\mathrm{e}^{-6it} + \tfrac{3i}{20}\bar{B}_4\mathrm{e}^{4it}, \\ & -2iA_1 t - \tfrac{1}{2}(B_1 - \tfrac{9}{2}\bar{A}_3)\mathrm{e}^{-2it} \\ & - \tfrac{1}{2}(\bar{B}_1 + \tfrac{3}{2}A_3)\mathrm{e}^{2it} - \tfrac{1}{4}B_3\mathrm{e}^{-4it} - \tfrac{1}{4}\bar{B}_3\mathrm{e}^{4it}\big), \end{aligned} \tag{107}$$

$$\begin{aligned} \mathbf{p}_1(t) = \big(& A_1\mathrm{e}^{it} + B_1\mathrm{e}^{-it} + \tfrac{3}{2}A_3\mathrm{e}^{3it} + \tfrac{3}{4}B_3\mathrm{e}^{-3it} - \tfrac{1}{4}\bar{B}_3\mathrm{e}^{5it}, \\ & i(\bar{B}_1 - 3A_3)\mathrm{e}^{it} - iA_1\mathrm{e}^{-it} - \tfrac{i}{4}B_3\mathrm{e}^{-5it} + \tfrac{3i}{4}\bar{B}_3\mathrm{e}^{3it} - \tfrac{3i}{2}\bar{A}_3\mathrm{e}^{-3it}, \\ & -2A_2\mathrm{e}^{it} + 2\bar{A}_2\mathrm{e}^{-it} - \tfrac{2}{3}(B_2 - 4\bar{A}_4)\mathrm{e}^{-3it} \\ & - \tfrac{2}{3}(\bar{B}_2 + 2A_4)\mathrm{e}^{3it} - \tfrac{3}{5}B_4\mathrm{e}^{-5it} - \tfrac{3}{5}\bar{B}_4\mathrm{e}^{5it}\big), \end{aligned} \tag{108}$$

$$\begin{aligned}\mathbf{p}_2(t) = \bigl(&A_2\mathrm{e}^{2it} + B_2\mathrm{e}^{-2it} + 2A_4\mathrm{e}^{4it} + \tfrac{6}{5}B_4\mathrm{e}^{-4it} - \tfrac{3}{10}\bar{B}_4\mathrm{e}^{6it}, \\ &i(\bar{B}_2 - 2A_4)\mathrm{e}^{2it} - i\bar{A}_2\mathrm{e}^{-2it} - \tfrac{3i}{10}B_4\mathrm{e}^{-6it} + \tfrac{6i}{5}\bar{B}_4\mathrm{e}^{4it} - 2i\bar{A}_4\mathrm{e}^{-4it}, \\ &-\tfrac{3}{2}A_3\mathrm{e}^{2it} + \tfrac{3}{2}\bar{A}_3\mathrm{e}^{-2it} - \tfrac{3}{4}B_3\mathrm{e}^{-4it} - \tfrac{3}{4}\bar{B}_3\mathrm{e}^{4it}\bigr),\end{aligned} \tag{109}$$

$$\begin{aligned}\mathbf{p}_3(t) = \bigl(&A_3\mathrm{e}^{3it} + B_3\mathrm{e}^{-3it}, i\bar{B}_3\mathrm{e}^{3it} - i\bar{A}_3\mathrm{e}^{-3it}, \\ &-\tfrac{4}{3}A_4\mathrm{e}^{3it} + \tfrac{4}{3}\bar{A}_4\mathrm{e}^{-3it} - \tfrac{4}{5}B_4\mathrm{e}^{-5it} - \tfrac{4}{5}\bar{B}_4\mathrm{e}^{5it}\bigr),\end{aligned} \tag{110}$$

$$\mathbf{p}_4(t) = \bigl(A_4\mathrm{e}^{4it} + B_4\mathrm{e}^{-4it}, i\bar{B}_4\mathrm{e}^{4it} - i\bar{A}_4\mathrm{e}^{-4it}, 0\bigr), \tag{111}$$

$$\mathbf{q}_1(t) = (\mathrm{e}^{it}, i\mathrm{e}^{-it}, 0) \qquad \textit{and} \qquad \mathbf{q}_2(t) = (0,0,1), \tag{112}$$

where $A_1 \in \mathbb{R}$ and $A_2, A_3, A_4, B_1, \ldots, B_4 \in \mathbb{C}$. Define a subset N of $\mathbb{C}^3$ to be

$$\bigl\{\mathbf{p}_0(t) + x\,\mathbf{p}_1(t) + \cdots + x^4\,\mathbf{p}_4(t) + y\,\mathbf{q}_1(t) + xy\,\mathbf{q}_2(t) : x, y, t \in \mathbb{R}\bigr\}. \tag{113}$$

Then N is a special Lagrangian 3-fold in $\mathbb{C}^3$ wherever it is nonsingular.

The expression (107) for $\mathbf{p}_0(t)$ could also include three complex constants of integration, but we have set them to zero for simplicity. For general $k \geqslant 1$, one can prove the following result.

Theorem 12.6 *In the situation above, for each $k \geqslant 1$ there exist solutions a_j, b_j, c_j to equations (90)–(95) depending on $A_1 \in \mathbb{R}$, $A_2, \ldots, A_k \in \mathbb{C}$ and $B_1, \ldots, B_k \in \mathbb{C}$, such that*

- (i) *For $1 \leqslant j \leqslant k$, a_j is a real linear combination of terms $A_{j+2l}\mathrm{e}^{i(j+2l)t}$, $B_{j+2l}\mathrm{e}^{-i(j+2l)t}$, $\bar{A}_{j+2l+4}\mathrm{e}^{-i(j+2l+2)t}$ and $\bar{B}_{j+2l+2}\mathrm{e}^{i(j+2l+4)t}$;*
- (ii) *For $1 \leqslant j \leqslant k$, b_j is a real linear combination of terms $iA_{j+2l+2}\mathrm{e}^{i(j+2l)t}$, $iB_{j+2l+2}\mathrm{e}^{-i(j+2l+4)t}$, $i\bar{A}_{j+2l}\mathrm{e}^{-i(j+2l)t}$ and $i\bar{B}_{j+2l}\mathrm{e}^{i(j+2l)t}$;*
- (iii) *For $1 \leqslant j \leqslant k$, c_j is a real linear combination of terms $A_{j+2l+1}\mathrm{e}^{i(j+2l)t}$, $B_{j+2l+1}\mathrm{e}^{-i(j+2l+2)t}$, $\bar{A}_{j+2l+1}\mathrm{e}^{-i(j+2l)t}$ and $\bar{B}_{j+2l+1}\mathrm{e}^{i(j+2l+2)t}$;*
- (iv) *a_0 is a real linear combination of terms $i\bar{A}_2 t$, $A_{2l+2}\mathrm{e}^{i(2l+2)t}$, $B_{2l+2}\mathrm{e}^{-i(2l+2)t}$, $\bar{A}_{2l+4}\mathrm{e}^{-i(2l+2)t}$ and $\bar{B}_{2l+2}\mathrm{e}^{i(2l+4)t}$;*
- (v) *b_0 is a real linear combination of terms $A_2 t$, $iA_{2l+4}\mathrm{e}^{i(2l+2)t}$, $iB_{2l+2}\mathrm{e}^{-i(2l+4)t}$, $i\bar{A}_{2l+2}\mathrm{e}^{-i(2l+2)t}$ and $i\bar{B}_{2l+2}\mathrm{e}^{i(2l+2)t}$; and*
- (vi) *c_0 is a real linear combination of terms $iA_1 t$, $A_{2l+3}\mathrm{e}^{i(2l+2)t}$, $B_{2l+1}\mathrm{e}^{-i(2l+2)t}$, $\bar{A}_{2l+3}\mathrm{e}^{-i(2l+2)t}$ and $\bar{B}_{2l+1}\mathrm{e}^{i(2l+2)t}$.*

Here in each case $l \geqslant 0$ is an integer, and we take $A_j = B_j = 0$ for $j > k$.

One surprising thing about this theorem is that the solutions contain no terms like $t^a\mathrm{e}^{ibt}$ for $a > 0$ and b integers, except for the t terms in a_0, b_0 and c_0. Here is the reason why. Let $1 \leqslant j < k$, and suppose by induction that we know a_i, b_i, c_i for $i = j+1, \ldots, k$, and that parts (i)–(iii) hold for them. To find a_j we must solve equation (96). By part (iii) above, the right hand side of (96) is a linear combination of exponentials e^{int} for various integers n.

The corresponding terms in a_j are multiples of e^{int} if $n \neq \pm j$, but multiples of $t\mathrm{e}^{int}$ if $n = \pm j$. Thus for a_j to be of the form given in part (i), we need the right hand side of (96) to contain no multiples of $\mathrm{e}^{\pm ijt}$. By part (iii), we expect to get a multiple of $\bar{A}_{j+2}\mathrm{e}^{-ijt}$. However, this multiple is zero, as we saw in (101) above, and so a_j satisfies (i). It easily follows that b_j and c_j satisfy (ii) and (iii), and the inductive step is complete.

12.4 Periodicity

In Theorems 12.5 and 12.6, the only terms which are not periodic in t with common period 2π are those in $\bar{A}_2 t$, $A_2 t$ and $A_1 t$ in equation (107) and parts (iv)–(vi) of Theorem 12.6. So setting $A_1 = A_2 = 0$ gives $\Phi(x,y,t+2\pi) = \Phi(x,y,t)$ for all x,y,t, where Φ is defined in (88).

Furthermore, Theorem 12.6 shows that $\mathbf{p}_j(t+\pi) = (-1)^j \mathbf{p}_j(t)$, and clearly $\mathbf{q}_j(t+\pi) = (-1)^j \mathbf{q}_j(t)$. This gives $\Phi(x,y,t+\pi) = \Phi(-x,-y,t)$ for all $x,y,t \in \mathbb{R}$. We may therefore regard Φ as mapping $\mathbb{R}^3/\mathbb{Z} \to \mathbb{C}^3$, where $\mathbb{Z}$ acts on $\mathbb{R}^3$ by $(x,y,t) \xmapsto{n} \bigl((-1)^n x, (-1)^n y, y + n\pi\bigr)$, for $n \in \mathbb{Z}$.

For $k \geqslant 3$ one can show that Φ is an immersion for generic values of $A_3, \ldots, B_k$. Then N is a nonsingular immersed 3-submanifold diffeomorphic to $\mathbb{R}^3/\mathbb{Z}$, or equivalently to $\mathcal{S}^1 \times \mathbb{R}^2$. We have proved:

Theorem 12.7 *For each $k \geqslant 3$, the construction above with $A_1 = A_2 = 0$ gives a family of special Lagrangian 3-folds in $\mathbb{C}^3$ depending on $A_3, \ldots, A_k$ and $B_1, \ldots, B_k \in \mathbb{C}$, with generic member a closed, nonsingular, immersed 3-submanifold diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$.*

Here is a parameter count for this family. It depends upon $A_3, \ldots, A_k$ and $B_1, \ldots, B_k \in \mathbb{C}$, which is $4k-4$ real parameters. It can be shown that the only symmetry left to take into account is translation in time, $t \mapsto t + c$. Subtracting one for this, the family of distinct SL 3-folds in $\mathbb{C}^3$ in Theorem 12.7, up to automorphisms of $\mathbb{C}^3$, has dimension $4k-5$.

Next we discuss the *asymptotic behaviour* of the SL 3-folds N of Theorem 12.7 near infinity in $\mathbb{C}^3$. Suppose for simplicity that A_k and B_k are not both zero, since if they are we can reduce k to $k-1$. Then $\mathbf{p}_k$ is nonzero for all t, by (97). It is clear from (88) that $\Phi(x,y,t) \to \infty$ in $\mathbb{C}^3$ as $(x,y) \to \infty$ in $\mathbb{R}^2$. To describe the asymptotic behaviour at infinity to leading order, we need to decide which terms in (88) are dominant when x, y are large, and neglect the other terms.

Obviously, when $|x|$ is large the term $x^k\,\mathbf{p}_k(t)$ dominates the terms $x^j\,\mathbf{p}_j(t)$ for $j < k$, as $p_k(t)$ is always nonzero. Thus the three terms in (88) which may dominate are $x^k\,\mathbf{p}_k(t)$, $y\,\mathbf{q}_1(t)$ and $xy\,\mathbf{q}_2(t)$. Careful thought shows that for fixed t there are the following four different asymptotic regimes, depending on the relative sizes of x and y:

(i) $x \gg 0$, $x = O(r^{1/k})$, $y = O(r^{(k-1)/k})$, with $\Phi(x,y,t) \approx x^k\,\mathbf{p}_k(t) + xy\,\mathbf{q}_2(t)$.

(ii) $y \gg 0$, $x = O(1)$, $y = O(r)$, with $\Phi(x,y,t) \approx y\,\mathbf{q}_1(t) + xy\,\mathbf{q}_2(t)$.

(iii) $x \ll 0$, $x = O(r^{1/k})$, $y = O(r^{(k-1)/k})$, with $\Phi(x,y,t) \approx x^k\,\mathbf{p}_k(t) + xy\,\mathbf{q}_2(t)$.

(iv) $y \ll 0$, $x = O(1)$, $y = O(r)$, with $\Phi(x,y,t) \approx y\,\mathbf{q}_1(t) + xy\,\mathbf{q}_2(t)$.

Here r is the radius function on $\mathbb{C}^3$.

These four regimes join onto each other in a cyclic fashion, with the single dominant term $xy\,\mathbf{q}_2(t)$ at the junction, so that for instance the junction between (i) and (ii) we have $x \gg 0$, $y \gg 0$ and $\Phi(x,y,t) \approx xy\,\mathbf{q}_2(t)$. When t varies as well we can identify (i) with (iii), and (ii) with (iv), since $\Phi(x,y,t+\pi) = \Phi(-x,-y,t)$. So there are only really two kinds of asymptotic behaviour to consider.

Using the expressions above for $\mathbf{p}_k$ and $\mathbf{q}_2$, we find that in case (i) we have

$$\Phi(x,y,t) \approx \bigl(x^k(A_k e^{ikt} + B_k e^{-ikt}), x^k(i\bar{B}_k e^{ikt} - i\bar{A}_k e^{-ikt}), xy\bigr),$$

with $x > 0$. This sweeps out the special Lagrangian 3-plane

$$L_1 = \bigl\langle (A_k + B_k, i\bar{B}_k - i\bar{A}_k, 0), (-iA_k + iB_k, \bar{B}_k + \bar{A}_k, 0), (0,0,1) \bigr\rangle_{\mathbb{R}}$$

in $\mathbb{C}^3$. Considering t to be a cyclic coordinate with period 2π, we see that to leading order Φ is a *k-fold branched cover* of L_1, branched along the real line $\bigl\langle (0,0,1) \bigr\rangle_{\mathbb{R}}$.

Similarly, in case (ii) we have

$$\Phi(x,y,t) \approx (y e^{it}, iy e^{-it}, xy),$$

with $y > 0$. This sweeps out the special Lagrangian 3-plane

$$L_2 = \bigl\langle (1,i,0), (i,1,0), (0,0,1) \bigr\rangle_{\mathbb{R}}$$

in $\mathbb{C}^3$. As t is cyclic with period 2π this is a 1-1 correspondence, rather than a branched cover. If $A_k = 0$ then $L_1 = L_2$. We shall assume $A_k \neq 0$ for simplicity, though it doesn't make much difference.

Thus we arrive at the following description of the SL 3-fold N at infinity in $\mathbb{C}^3$. It is weakly asymptotic to the union of two special Lagrangian 3-planes L_1, L_2 in $\mathbb{C}^3$, which intersect in the real line $\bigl\langle (0,0,1) \bigr\rangle_{\mathbb{R}}$. Along L_1 it converges to a k-fold branched cover, branched along $\bigl\langle (0,0,1) \bigr\rangle_{\mathbb{R}}$, so that k points of N 'converge' to 1 point of L_1 at infinity. Along L_2 the convergence is 1-1.

The boundary of $N \cong \mathcal{S}^1 \times \mathbb{R}^2$ at infinity is T^2, whereas the boundary of $L_1 \cup L_2$ at infinity is two copies of $\mathcal{S}^2$, intersecting in two points. The T^2 wraps itself round these two $\mathcal{S}^2$, so as to cover the first k times and the second once. The order of convergence is $O(r^{(k-1)/k})$, which is rather weak, as it means that towards infinity N gets further away from $L_1 \cup L_2$ rather than closer.

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